Towards the quantum geometry of the M5-brane in a constant C-field from multiple membranes

Towards the quantum geometry of the M5-brane in a constant C-field from multiple membranes

Chong-Sun Chu and Douglas J. Smith

Centre for Particle Theory and Department of Mathematics, Durham University,
Durham, DH1 3LE, U.K.

E-mail: chong-sun.chu@durham.ac.uk, douglas.smith@durham.ac.uk

ABSTRACT: We show that the Nahm equation which describes a fuzzy D3-brane in the presence of a B-field can be derived as a boundary condition of the F1-strings ending on the D3-brane, and that the modifications of the original Nahm equation by a B-field can be understood in terms of the noncommutative geometry of the D3-brane. Naturally this is consistent with the alternative derivation by quantising the open strings in the B-field background. We then consider a configuration of multiple M2-branes ending on an M5-brane with a constant 3-form C-field. By analogy with the case of strings ending on a D3-brane with a constant B-field, one can expect that this system can be described in terms of the boundary of the M2-branes moving within a certain kind of quantum geometry on the M5-brane worldvolume. By repeating our analysis, we show that the analogue of the B-field modified Nahm equation, the C-field modified Basu-Harvey equation can also be understood as a boundary condition of the M2-branes. We then compare this to the M5-brane BIon description and show that the two descriptions match provided we postulate a new type of quantum geometry on the M5-brane worldvolume. Unlike the D-brane case, this is naturally expressed in terms of a relation between a 3-bracket of the M5-brane worldvolume coordinates and the C-field.

KEYWORDS: Brane Dynamics in Gauge Theories, D-branes, Non-Commutative Geometry, M-Theory

ARXIV EPRINT: 0901.1847

Contents

1 Introduction and summary

One of the most interesting recent developments in string theory is the discovery of a description of the conformal field theory describing multiple membranes [1–4]. The Bagger-Lambert (BL) theory [1–3] was originally motivated by trying to construct an action with manifest $\mathcal{N} = 8$ superconformal symmetry, based on a BPS equation postulated by Basu and Harvey [5]. This naturally led to an action with a non-Abelian symmetry based on a 3-algebra. Unfortunately this is problematic as there is only one example of such a 3-algebra, describing 2 M2-branes [6]. Attempts have been made to circumvent this difficulty by relaxing some of the constraints on the 3-algebra. E.g. relaxing the requirement for the 3-algebra to have a Euclidean metric turns out to give a consistent action describing N branes [7], but it appears that this gives an alternative description of the low energy limit of N D2-branes rather than M2-branes [8]. However, it has been argued that the M2-brane theory can be described by a non-Abelian twisted Chern-Simons theory [9] which has manifest $\mathcal{N} = 6$ superconformal symmetry. This theory was not described using a 3-algebra, but it has been shown that there is an equivalent description using a type of 3-algebra which is not completely anti-symmetric [10]. It therefore seems that the 3-algebra

is not needed to describe multiple M2-branes, but it can be used and indeed can be viewed as a natural way to encode various constraints on the couplings and matter content of the theory.

The M5-brane is another mysterious object in M-theory. It is somewhat analogous to a D-brane in string theory, in the sense that M2-branes can end on it. However, it contains a self-dual 3-form field strength, rather than the 2-form field strength on D-branes. The self-duality of this field leads to technical complications, but the action and equations of motion for a single M5-brane are known [11–13]. On the other hand, the action or equations of motion for multiple M5-branes are not understood as it is not at all clear how to formulate a theory with a non-Abelian 3-form field strength. It would certainly be interesting to understand more about the theory of multiple M5-branes.

Since M2-branes can end on M5-branes, one may wonder what can be learned of the M5-brane from the M2-brane by considering such an intersection. It is known that the equations of motion for a single M5-brane can be derived by demanding the κ-symmetry of the open membrane ending on it [14]. It is also known that this intersecting configuration can be described in terms of the M5-brane theory as a BIon spike, with the M2-branes emerging from the M5-brane worldvolume. Alternatively, it can be described in terms of multiple M2-branes as a fuzzy funnel, with the extra 3 worldvolume dimensions of the M5-brane arising as a fuzzy 3-sphere. In fact this latter description is given by a solution of the Basu-Harvey equation which was consequently proposed as a BPS equation for the BL theory of multiple M2-branes.

In an analogous but simpler system with a string ending on a D-brane, one of the interesting results that can be obtained from the open string ending on the D-brane is that the D-brane worldvolume becomes noncommutative when there is a constant NS B-field present on the D-brane worldvolume [15–17]. The result can be derived by quantising the open string ending on the D-brane. Due to the B-field, the usual Neumann boundary condition becomes a mixed one. A proper quantization taking account of the boundary condition [18–20] gives the result that the endpoints of the open strings, in other words the D3-brane worldvolume coordinates X^i become noncommutative and obey the commutation relation:

$$\left[X^i, X^j \right] = i\theta^{ij}, \tag{1.1}$$

where θ^{ij} is an antisymmetric constant matrix whose components are related to the components of the NS B-field [18].

Naturally one would like to repeat the same steps for an M5-brane with a constant C-field on it and derive the form of quantum geometry on the M5-brane. In this case the boundary condition is also a mixed one, but is nonlinear in the boundary coordinates. The analysis is much more complicated due to the nonlinear nature of the membrane action and one can only do an approximate analysis. These results were expressed in terms of commutators of the boundary string coordinates $X^i(\tau, \sigma_1, \sigma_2 = 0)$ and look very complicated [21]. Also it is not clear how the full exact results may be obtained in a consistent manner from this approach.

In this paper we argue that the quantum geometry over the M5-brane naturally takes

the form

$$\left[X^i, X^j, X^k \right] = i\Theta^{ijk}, \tag{1.2}$$

where Θ^{ijk} is a constant completely antisymmetric matrix whose components are related to the components of the constant C-field. This suggests that the natural language to encode the quantum geometry for the M5-brane in the presence of a constant C-field is in terms of a 3-bracket rather than a commutator.

To start with, we recall the familiar analogous situation in string theory and re-derive the noncommutative geometry (1.1) using a new method. Our starting point is the observation that the D3-D1 intersecting system can be looked upon as a single brane where one kind of brane can be constructed as a solitonic configuration of the other system of branes: from the D3-brane point of view, the D1-branes can be described as a magnetic monopole of charge N in the Abelian Born-Infeld theory of the D3-brane [22]. In terms of the D1-strings theory, one can describe the D3-brane as a certain solution of the Nahm equation whose transverse scalar fields are described by a fuzzy two-sphere [23, 24]. The fuzzy dimensions provide the two extra dimensions to build up the D3-brane from the D1-strings. Note that in this dual description of the D1-D3 intersecting system, it is necessary to consider a large number N of D1-branes ending on a D3-brane in order for the description to be valid.

The construction of the D3-brane *out of* D1-branes is interesting. However since the original system we are describing is really a system of D1-strings ending on a D3-brane, it suggests that one should be able to understand the defining Nahm equation as a boundary condition of the open D1-strings. We show that this is indeed the case. The theory we are using here is a matrix theory of the D1-strings. This way to derive the Nahm equation thus provides a new way to understand the intersecting D1-D3 branes system. This is one of the main results of this paper.

One can also include a B-field in the transverse directions of the D1-strings, corresponding to having a B-field on the D3-brane. This system has been studied both as a BIon spike [25] or as a fuzzy funnel [27]. Basically the bunch of D1-branes was found to have a deformed shape and to be tilted away from the normal to the D3-brane. These features can be explained by a constant shift in the Nahm equation. This modified Nahm equation was derived as a BPS equation of the non-Abelian Born-Infeld theory of the D1-branes [27]. Using our new understanding of the Nahm equation, we re-derive this equation as a boundary condition of the theory of matrix F1-strings ending on the D3-brane. In this analysis, this shift can be understood precisely in terms of the noncommutative geometry (1.1) of the D3-brane. Conversely if one did not know about the noncommutative geometry of the D3-brane, one could have derived it this way by using the fact that the constant shift in the Nahm equation is known to be present by matching with the BIon description with B-field. The noncommutative geometry (1.1) was originally derived by quantising the open string in a background B-field. Our derivation of the Nahm equation in the presence of a B-field thus provides a new way to derive (1.1). This is another main result of the paper.

Our main interest is the M2-M5 intersecting system. Applying the same idea, we show that the Basu-Harvey equation can be derived as a boundary condition of the theory of multiple M2-branes. We then proceed to include a C-field on the M5 brane. The system

has been studied from the M5-brane point of view where the M2-branes bundle has been constructed as a certain static charge configuration protruding out of the M5-brane. Due to the C-field, the M2-branes are static only if tilted away from the normal to the M5-brane. The C-field also modifies the shape of the M2-branes funnel. It turns out one can reproduce precisely the tilting and the shape of the M2-branes funnel from the M5-brane point of view [29, 30] if the Basu-Harvey equation is modified in a particular way. We identify and propose this modified Basu-Harvey equation as a description of M5-branes with C-field.

Now just as one can understand the Nahm equation with B-field as a boundary condition of the theory of open F1-strings, and since we have already shown that the original Basu-Harvey equation can be understood as a boundary condition of open membranes, it strongly suggests that one should be able to understand the modified Basu-Harvey equation as a boundary condition of open membranes in the presence of C-field. We show that this is indeed the case provided that the M5-brane worldvolume is described by a quantum geometry of the form (1.2). This is another main result of the paper.

The plan of the paper is as follows. In section 2, we first review the dual descriptions of the D1-D3 intersecting system in terms of the BPS equations of the corresponding Born-Infeld theories: a monopole equation for the Abelian Born-Infeld theory for the D3-brane and a Nahm equation for the non-Abelian Born-Infeld theory for the D1-strings. In section 3, we show that the Nahm equation can be understood as a boundary condition of the matrix theory of D1-strings which end on the D3-brane. In section 4, we include a NS B-field. By using a system of F1-strings to probe the D1-D3 system, we show that the Nahm equation with B-field can be derived as a boundary condition of the matrix theory of F1-strings. In particular we show that the Nahm equation with B-field encodes the noncommutative geometry of the D3-brane as well as information about the open string metric [17] on the D3-brane. In section 5, we apply our idea to the M2-M5 intersecting system and derive the Basu-Harvey equation as a boundary condition of the theory of multiple open membranes. We then generalize the Basu-Harvey equation by including a constant C-field by checking that our modification results in a fuzzy-funnel configuration of M2-branes with an M5-brane at an angle, reproducing the known result from the M5-brane BIon solution in a constant C-field. We also show that this equation can be understood as the boundary condition of open membranes ending on the M5-brane with C-field provided that the quantum geometry (1.2) of the M5-brane holds. The paper concludes with some further discussions.

2 Review of dual descriptions of D1-D3-branes intersections

Consider a system of N D1-strings ending on a D3-brane. The intersecting brane system can be described in two different ways in terms of either the D3-brane theory or the D1-strings theory.

2.1 The bion solution of the D3-brane

From the D3-brane theory point of view, the Abelian Born-Infeld action is

$$S_{D3} = -T_3 \int d^4x \sqrt{-\det\left(\eta_{\mu\nu} + \partial_\mu X^A \partial_\nu X^A + \lambda(F_{\mu\nu} + B_{\mu\nu})\right)}, \tag{2.1}$$

where $\lambda = 2\pi\alpha'$ and

$$T_p = \frac{1}{g_s(2\pi)^p l_s^{p+1}} \tag{2.2}$$

is the tension for Dp-brane. Here $\mu, \nu = 0, 1, 2, 3$ are the worldvolume indices of the D3-brane, $A, B = 4, \cdots, 9$ are the indices of the transverse space, and $B_{\mu\nu}$ is the pull-back of the B-field to the worldvolume. One finds that the Born-Infeld theory supports solitonic configurations which describe D1-strings protruding from the D3-brane [22]. For a configuration with a single transverse excitation $X^9(x^i)$, the Born-Infeld theory (2.1) admits a static BPS equation of the form:

$$\partial_i X^9 = B_i, \quad i, j, k = 1, 2, 3, \tag{2.3}$$

where $B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}$ is the magnetic field on the D3-brane. This equation coincides with the BPS equation for magnetic monopoles. A simple solution is given by

$$X^9(x^i) = \frac{Q}{\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}}, \qquad B_i = -\frac{N}{2r^3}\vec{r} \tag{2.4}$$

where $Q := \pi\alpha'N$, N is an integer. This corresponds to placing N units of U(1) magnetic charge on the D3-brane and describes a spike of N D1-strings coming out of the D3-brane. It is interesting to note that at a fixed distance $X^9 = \sigma$ from the D3-brane, the cross-section of the D3-brane is a 2-sphere with radius

$$r(\sigma) = \frac{Q}{\sigma}. \tag{2.5}$$

One can also include a constant NS B-field on the worldvolume of the D3-brane. Let's take a B-field in the 12-direction, $B_{12} \neq 0$. Due to the B-field, the spherical symmetrical is broken. It was found that [25] the solution is modified to one whose cross section becomes an ellipsoid

$$\frac{x_1^2}{r_1(\sigma)^2} + \frac{x_2^2}{r_2(\sigma)^2} + \frac{(x_3 - \sigma\tan\alpha)^2}{r_3(\sigma)^2} = 1, \tag{2.6}$$

where

$$r_1(\sigma) = r_2(\sigma) = \frac{Q}{\sigma}\cos\alpha, \quad r_3(\sigma) = \frac{Q}{\sigma} \tag{2.7}$$

and

$$\tan\alpha = 2\pi\alpha'B. \tag{2.8}$$

Note that the ellipsoid is centred at the coordinates $(x^1, x^2, x^3) = (0, 0, \sigma\tan\alpha)$ and describes D1-strings tilted away from the normal to the D3-brane by an angle α.

2.2 The fuzzy funnel solution of the D1-branes

One can also consider the dual description and study the system from the D1-strings point of view. The Born-Infeld action is [26]

$$S_{D1} = -T_1 \int d^2\sigma \; \text{STr} \left[\sqrt{-\det\left(P_{ab}\left[E_{MN} + E_{MI}(Q^{-1} - \delta)^{IJ}E_{JN}\right] + \lambda\, F_{ab}\right) \det(Q^I{}_J)} \right], \tag{2.9}$$

where

$$E_{MN} = G_{MN} + B_{MN}, \qquad \text{and} \qquad Q^I{}_J := \delta^I{}_J + i\lambda \left[\Phi^I, \Phi^K\right] E_{KJ} \tag{2.10}$$

and Φ^i are scalar fields of mass dimension. The matrix transverse coordinates are defined by $X^i = 2\pi\alpha'\Phi^i$. Here $a, b = \sigma, \tau$ are the worldvolume indices of the D1-branes, $I, J, K = 1, \cdots, 8$ are the indices of the transverse space, and M, N are the ten-dimensional spacetime indices. As was shown in [23, 24], the static solution of the non-Abelian Born-Infeld theory of the D1-branes satisfies the Nahm equation

$$\partial_\sigma \Phi_i = i\frac{1}{2}\epsilon_{ijk}\left[\Phi^j, \Phi^k\right], \tag{2.11}$$

where (τ, σ) are the worldsheet coordinates of the D1-branes and $\epsilon_{123} = 1$. The solution $\Phi = 0$ corresponds to an infinitely long bundle of coincident D1-branes. In [23], another solution was found by allowing a singular boundary condition[1] at $\sigma = 0$ (see (2.13) below). This new solution describes a fuzzy 2-sphere

$$\Phi^i(\sigma) = f(\sigma)\alpha^i, \tag{2.12}$$

where α^i form an $N \times N$ representation of the generators of an SU(2) subgroup of SU(N), $[\alpha^i, \alpha^j] = 2\epsilon^{ijk}\alpha^k$ and f is given by

$$f = \frac{1}{2\sigma}. \tag{2.13}$$

Note that Φ diverges at $\sigma = 0$. As emphasised in [23], this new feature was essential to their construction.

This fuzzy funnel solution carries the correct RR charge and tension to be identified with an orthogonal D3-brane. It also matches nicely with the BIon solution in the large N limit. To see this, we note that at a fixed point σ on the D1-branes, the geometry is that of a fuzzy sphere with radius R given by

$$R^2 = \frac{1}{N}\text{tr}\left(X^i\right)^2. \tag{2.14}$$

This gives for the above solution

$$R(\sigma) = \frac{\pi\alpha'\sqrt{N^2 - 1}}{|\sigma|} \approx \frac{\pi\alpha'N}{|\sigma|} \tag{2.15}$$

---

[1] This singular boundary condition was first discussed in the context of D-branes in [31]. Some other boundary conditions were considered in [32]. We emphasis that these are a different kind of boundary condition from what we are going to derive in the next section.

for large N. This matches precisely with the relation (2.5) derived above for the BIon solution.

One may also add a NS B-field in the spatial directions transverse to the D1-brane. For a constant B field with $B_{12} := B \neq 0$, the effect of the B-field to the D1-brane Born-Infeld action was found to modify the Nahm equation [27] to

$$\partial_\sigma \phi^i = i \left(\frac{1}{2} \epsilon_{ijk} \left[\phi^j, \phi^k \right] + \delta_i^3 iB \right), \tag{2.16}$$

where the rescaled fields ϕ^i are defined by

$$\phi^1 := \sqrt{1 + (2\pi\alpha'B)^2}\Phi^1, \quad \phi^2 := \sqrt{1 + (2\pi\alpha'B)^2}\Phi^2, \quad \phi^3 := \Phi^3. \tag{2.17}$$

The modified Nahm equation has the solution

$$\phi^i = f(\sigma)\alpha^i - \delta_3^i \, B \, \sigma, \tag{2.18}$$

where f is the same as (2.13) above. To compare with the BIon solution described in section 2.1, one should go the description in terms of X^i. Using the scaling (2.17), the solution (2.18) becomes a fuzzy ellipsoid with radii (2.7) and a tilting given precisely as in (2.8).

Before we close this section, it is instructive to rewrite the Nahm equation (2.16) in terms of the physical variables X^i:

$$\partial_\sigma X^i = \frac{i}{\lambda} \epsilon^{ij} \left[X^j, X^3 \right], \quad i, j = 1, 2,$$
$$\partial_\sigma X^3 = \frac{i}{\lambda} \left(1 + \left(2\pi\alpha'B \right)^2 \right) \left(\left[X^1, X^2 \right] + i\theta \right), \tag{2.19}$$

where θ is the constant

$$\theta := \frac{2\pi\alpha'B}{1 + (2\pi\alpha'B)^2}. \tag{2.20}$$

In the following two sections we will show that the (modified) Nahm equation can be derived as a boundary condition for the open D1-strings or F1-strings ending on a D3-brane. In particular the understanding of the Nahm equation as boundary condition of the F1-strings will provide us a understanding to the modification of the Nahm equation (2.19) in the presence of a constant B-field in terms of the noncommutative geometry of the D3-brane.

3 Nahm equation as boundary condition of D1-strings

In this section we show how to derive the Nahm equation as a boundary condition of the matrix theory of D1-strings. Let us start by writing down the action of the matrix theory of D1-strings. This can be derived by taking the Yang-Mills ($\alpha' \to 0$) limit of the Born-Infeld action (2.9) for the D1-strings. The fluctuations give the matrix string action (the bosonic part)

$$S_{D1} = -\frac{1}{\lambda g_s} \int d^2\sigma \text{tr} \left(\frac{1}{2} \left(\partial X^I \right)^2 - \frac{1}{4\lambda^2} \left[X^I, X^J \right]^2 + \frac{\lambda^2}{4} F_{ab}^2 \right), \tag{3.1}$$

where $X^I = \lambda \Phi^I$ and $I, J = 1, \cdots, 8$. Since D1-strings are magnetic sources for the Abelian gauge field on the D3-brane, they couple to the dual Abelian gauge field through the boundary coupling

$$\frac{1}{g_s} \int d\tau \ \mathrm{tr} \left[\tilde{A}_\mu(x^\nu(\tau)) \frac{dX^\mu(\tau)}{d\tau} \right], \tag{3.2}$$

where

$$x^\nu(\tau, \sigma) := \frac{1}{N} \mathrm{tr} X^\nu(\tau, \sigma) \tag{3.3}$$

is the center of mass coordinate of the matrix string coordinates and the sum is over $\mu = 0, 1, \cdots 3$. We will comment more on this coupling in section 4.2. Here $\tilde{F} := \mathrm{d}\tilde{A}$ is the Hodge dual of the field strength $F := dA$. In general, under S-duality, F is transformed to $S[F]$, which in the leading order is $S[F] = \tilde{F}$.

With this coupling, the boundary conditions for the fields X^I of the D1-strings are ($\sigma = 0$):

$$X^{i'} = \text{fixed}, \qquad\qquad i' = 4, \cdots, 8, \tag{3.4}$$

$$\frac{1}{2\pi\alpha'} \partial_\sigma X^i = \tilde{F}^{0i} + \tilde{F}_j{}^i \partial_\tau X^j, \qquad i = 1, 2, 3, \tag{3.5}$$

where $\tilde{F}$ is the field strength of the gauge field $\tilde{A}$. (3.5) can be written in a more illuminating way as

$$\frac{1}{2\pi\alpha'} \left(\partial_\sigma X^i + 2\pi\alpha' C^i{}_j \partial_\tau X^j \right) = \tilde{F}^{0i}, \tag{3.6}$$

where $C_{ij} := \tilde{F}_{ij}$ corresponds to a RR 2-form potential and the l.h.s. is simply the mixed boundary condition one would obtain for the D1-strings if there is a RR 2-form. Since there is only a NS B-field in our background, we will consider the case that $\tilde{F}_{ij} = 0$ and the boundary condition (3.5) reduces to

$$\frac{1}{2\pi\alpha'} \partial_\sigma X^i = \tilde{F}^{0i}. \tag{3.7}$$

Note that the electric field $\tilde{F}^{0i}$ is by definition equal to the magnetic field $B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}$. With the coordinate dependence explicitly spelt out, the boundary condition (3.5) for the D1-strings can thus be written as

$$\frac{1}{2\pi\alpha'} \partial_\sigma X^i(\sigma = 0) = \frac{1}{2}\epsilon_{ijk}F^{jk}\left(x^1, x^2, x^3\right) \quad i, j, k = 1, 2, 3, \tag{3.8}$$

where $x^i = \mathrm{tr}(X^i(\sigma = 0))/N$ are the coordinates of the D1-strings endpoint on the D3-brane. This is the equation we are interested in. Since this equation tells us about the presence of the D3-brane and its properties, one is tempted to think of (3.8) as the defining equation for the D3-brane. Note that since F_{ij} is a singlet in the U(N) gauge group of the D1-strings theory, $X^i(\sigma = 0) = x^i \mathbf{1}$ and (3.8) becomes a simple condition on the boundary coordinates of the center of mass string.

However, from the very general spirit of matrix model, one would expect to be able to construct the higher dimensional D3-brane from the D1-strings by exciting the transverse scalar fields of the D1-strings theory. In figure 1a, the circle denotes the endpoint of the
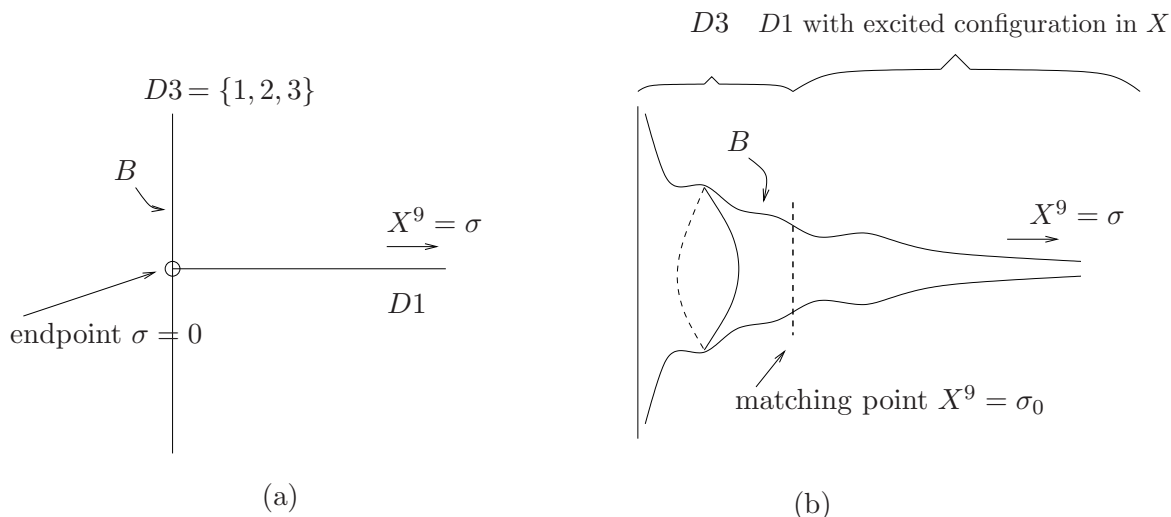
Figure 1. Boundary condition of D1-strings ending on D3-brane.

open string where the equation (3.8) holds. By exciting the transverse scalars on the D1-strings, the D3-D1 system becomes something like in figure 1b. It is somewhat arbitrary whether one would like to interpret the transverse scalars X^i as describing the geometry (in a generalized sense) of the *D3-brane* or as a scalar field configuration of the *D1-branes* theory. Let us consider a dividing point at $X^9 = \sigma_0 > 0$ and interpret the region on the l.h.s. of it as describing a D3-brane, and the region on the r.h.s. as describing the D1-strings, then the boundary condition (3.8) becomes

$$\frac{1}{2\pi\alpha'}\partial_\sigma X^i(\sigma_0) = \frac{1}{2}\epsilon_{ijk}F^{jk}\left(X^1, X^2, X^3\right),\qquad(3.9)$$

where $X^i = X^i(\sigma_0)$ denotes the matrix coordinates of the D3-brane and describes the "geometry" of the D3-brane. At the same time F_{ij} also becomes an $N \times N$ matrix.

Perhaps the simplest way to interpret (3.9) is to consider N widely separated D1-strings. Each would have a boundary condition of the form of (3.8)

$$\frac{1}{2\pi\alpha'}\partial_\sigma X^i = \frac{1}{2}\epsilon_{ijk}F^{jk}\left(x^1, x^2, x^3\right)\quad i, j, k = 1, 2, 3,\qquad(3.10)$$

where X^i are the coordinates of the D1-string and $x^i = X^i(\sigma = 0)$ are the coordinates of the D1-string endpoint on the D3-brane. Now, in general we have an $N \times N$ matrix description of the D1-strings coordinates, and we would expect to recover these N individual boundary conditions only for configurations where we could simultaneously diagonalise the matrices. Therefore, in general we expect a matrix equation of the form

$$\frac{1}{2\pi\alpha'}\partial_\sigma X^i = \frac{1}{2}\epsilon_{ijk}F^{jk}\quad i, j, k = 1, 2, 3,\qquad(3.11)$$

where F^{ij} is an $N \times N$ matrix encoding the value of the D3-brane U(1) field strength at the endpoints of the N D1-strings, whose positions are given by the eigenvalues of

the $N \times N$ matrices X^i. The matrices F^{ij} can be viewed as arising from a standard matrix discretisation of two worldvolume coordinates of the D3-branes, transforming the functional dependence of the fields on these coordinates into the discrete labels of the resulting matrices. When we further consider the situation at large N illustrated in figure 1b where the D1-strings actually generate the D3-brane, the D3-brane U(1) field strength, and hence the $N \times N$ matrix F^{ij} in (3.11) must be constructed from the fields in the D1-strings action. We therefore arrive at (3.9). Note that this construction of the D3-brane could alternatively be viewed as a specific discretisation of the D3-brane.

Now, the precise form of $F^{ij}(X)$ can be fixed by requiring that the action of a configuration supported by X^i in the matrix string theory should match with that of the corresponding YM terms of the D3-brane action

$$S_{D3} = -T_3 \int d^4x \left(\frac{1}{2} \left(\partial_\mu X^A \right)^2 + \frac{\lambda^2}{4} F_{\mu\nu}^2 \right) \tag{3.12}$$

in the large N limit. Doing so requires that

$$F^{ij} = \frac{i}{(2\pi\alpha')^2} \left[X^i, X^j \right], \quad i,j = 1,2,3 \tag{3.13}$$

and

$$\int d^4x = (2\pi l_s)^2 \int d^2\sigma \, \mathrm{tr} \tag{3.14}$$

where we have used $T_1/T_3 = (2\pi l_s)^2$.

With this identification, the boundary condition for the matrix theory of D1-strings reads:

$$\partial_\sigma \Phi^i = \frac{i}{2} \epsilon_{ijk} \left[\Phi^j, \Phi^k \right], \tag{3.15}$$

at $\sigma = \sigma_0$. However since the dividing point σ_0 is completely arbitrary, we conclude that the D1-D3 system depicted in figure 1b can be described by (3.15) for arbitrary $\sigma \geq 0$. Amazingly this is precisely the Nahm equation (2.11) derived from the Born-Infeld theory of the D1-strings. Here we have derived it as a boundary condition of the D1-strings matrix model.

We remark that the identification (3.13) is derived in the large N limit where the descriptions of the D1-D3 system in terms of the D3-brane Born-Infeld theory and the D1-branes Born-Infeld theory completely overlap [23, 24]. For finite N, there will be higher derivative corrections to the Born-Infeld actions. In general the two descriptions overlap only for a range of σ which depends on N and the identification (3.13) holds in this range. Outside this range, the identification (3.13) and equation (3.15) will receive $1/N$ corrections.

One can repeat the above analysis to include a B-field in the transverse directions of the D1-strings. Given the non-Abelian Born-Infeld action (2.9) for the D1-strings, one can derive how the B-field modifies the matrix string action. We obtain the following relevant terms in the D1-strings action:

$$S_{D1} = -\frac{1}{\lambda g_s} \int \frac{d^2\sigma}{\sqrt{1+\lambda^2 B^2}} \, \mathrm{tr} \left[\frac{1}{2}(\partial\phi^i)^2 - \frac{\lambda^2}{2} \left(\left[\phi^1,\phi^3\right]^2 + \left[\phi^2,\phi^3\right]^2 + \left(\left[\phi^1,\phi^2\right] + iB\right)^2 \right) + \cdots \right]. \tag{3.16}$$

Identifying the scalar interaction terms in the D1-strings action with F^2 terms in the D3-brane YM action derived from (2.1) with $B := B_{12} \neq 0$:

$$S_{D3} = -T_3 \int \frac{d^4 x}{\sqrt{1 + \lambda^2 B^2}} \left[\frac{\lambda^2}{2} \left(F_{13}^2 + F_{23}^2 + F_{12}^2 \right) + \cdots \right], \qquad (3.17)$$

we need (3.14) and

$$F^{ij} = i \left[\phi^i, \phi^j \right] - B \epsilon^{3ij}, \qquad (3.18)$$

which is a modified version of the identification (3.13). As we will argue below, the boundary condition in the presence of the B-field should take the form

$$\partial_\sigma \phi^i = \frac{1}{2} \epsilon_{ijk} F^{jk}. \qquad (3.19)$$

Therefore we obtain precisely the modified Nahm equation (2.16) when (3.18) is substituted into (3.19). This is how the Nahm equation gets modified by the B-field from the point of view of the D1 matrix string theory.

We note that the boundary condition (3.19) can be obtained from (3.16) by including the boundary coupling

$$\frac{1}{g_s \sqrt{1 + \lambda^2 B^2}} \int d\tau \, \text{tr} \left[\tilde{A}_i(x^\nu(\tau)) \frac{d\phi^i(\tau)}{d\tau} \right] \qquad (3.20)$$

in the gauge $\tilde{A}_0 = 0$. This coupling can be seen by the following argument. We note that (3.16) has an SO(3) invariance if we transform the fields ϕ^i and the external field B_{ij} as a triplet of SO(3). The boundary coupling should therefore respect the same symmetry. This is so because, due to the tensor gauge symmetry which keeps $B_{ij} + F_{ij}$ invariant, one can see that $\epsilon_{ijk} \tilde{F}^{0k}$, i.e. $\tilde{A}^i$ transforms in the same way as a triplet of SO(3). As for the overall factor in (3.20), we note that a NS B-field with only $B_{12} \neq 0$ should affect the magnetically charged D1 endpoint only in the 3rd direction. This fixes the overall coefficient since $\phi^{1,2} = \sqrt{1 + \lambda^2 B^2} X^{1,2}$.

We remark that a similar analysis cannot be carried through to the M2-M5 system since a Born-Infeld type action for multiple M2-branes which includes the effect of the C-field is not available.[2] Therefore to derive the C-field modified Basu-Harvey equation as a boundary condition of M2-branes, we will have to follow a different route.

In the next section we will provide another new way to understand the Nahm equation: as boundary condition of the system of F1-strings probing the D1-D3 system. Technically this approach has the advantage that the coupling of the NS B-field to the F1-strings is much simpler than that to the D1-strings. Conceptually, this approach allows us to understand this shift in the Nahm equation as the noncommutative geometry of the D3-brane in the presence of a B-field. We will see that this approach can be applied directly to the M2-M5 system and allows us to derive the form of the quantum geometry over the M5-brane in the presence of a C-field.

---

[2] See [33] for some results on nonlinear generalization of the BL theory, but note that these results do not contain a C-field.

4 Nahm equation from boundary condition of F1-strings

4.1 Matrix theory of IIB strings

According to the proposal of Banks, Fischler, Shenker and Susskind [35], the full M-theory in the infinite momentum frame is described by the large N limit of a U(N) supersymmetric quantum mechanics of D0-particles where N is the number of D0-particles. Compactification of the matrix theory on a spatial circle yields a description of the IIA superstrings by a two-dimensional supersymmetric Yang-Mills theory [36–38]. In particular, by performing a 9-11 flip, Dijkgraaf, Verlinde and Verlinde [38] obtained a description of the type IIA string theory in the lightcone gauge in terms of a $(1 + 1)$-dimensional U(N) Yang-Mills theory with $\mathcal{N} = 8$ supersymmetry

$$S = \frac{1}{2\pi\alpha'} \int \mathrm{tr}\left(\frac{1}{2}\left(D_\mu X^I\right)^2 + \pi^2 g_s^2 l_s^4 F_{\mu\nu}^2 - \frac{1}{4\pi^2 g_s^2 l_s^4}\left[X^I, X^J\right]^2 \right.$$
$$\left. + \theta^T \not{D}\,\theta + \frac{1}{2\pi g_s l_s^2}\theta^T \Gamma_I\left[X^I, \theta\right]\right). \tag{4.1}$$

Here the 8 scalar fields X^I, $I = 1\cdots,8$ and the 8 fermions fields θ_L^α, $\theta_R^{\dot\alpha}$ are $N \times N$ Hermitian matrices. The fields $X^I, \theta^\alpha, \theta^{\dot\alpha}$ transform respectively in the $\mathbf{8}_v$ vector, and $\mathbf{8}_S$ and $\mathbf{8}_c$ spinor representations of the SO(8) R-symmetry group.

As usual, the spacetime coordinates of the strings are identified with the eigenvalues of the X^I. The integral is over a cylindrical $(1 + 1)$-dimensional space, with coordinates τ and $\sigma \sim \sigma + 2\pi$. The fields do not necessarily have the same periodicity since they can be identified up to a gauge transformation under this shift in σ. Such a gauge transformation can permute the eigenvalues, leading to the interpretation of the configuration as a collection of closed strings, each with some periodicity $\sigma \sim \sigma + 2\pi n$, where this integer n also determines the lightcone momentum of this string. Hence the integer N determines the total lightcone momentum, not the number of closed strings.

The Yang-Mills coupling is dimensionful and is related to the string coupling and the string length as

$$g_{\mathrm{YM}} = \frac{1}{g_s l_s}. \tag{4.2}$$

It was shown [38] that in the IR limit of the SYM theory, the theory become strongly coupled and the diagonal components of the $N \times N$ matrices decouple from the off-diagonal components in this $g_s \to 0$ limit. Moreover, it was shown that the IR dynamics of the diagonal elements is described by a sigma model with orbifold target space $(R^8)^N/S_N$ and the Hilbert space of quantum states of this conformal field theory is the same as those for the second quantized type IIA superstring theory.

A T-duality relates the theory of IIA strings with the theory of IIB strings, where the same action (modulo a different chirality assignment of the spinors) now describes IIB strings. In this description, the action (4.1) with the spinors $\theta^\alpha, \theta^{\dot\alpha}$ being both $\mathbf{8}_S$ of SO(8) describes IIB strings stretched in the 9-th direction. Another way to derive this result is to note that under T-duality (i.e. instead of doing the 9-11 flip), the compactified BFSS

matrix theory describes a theory of D1-strings. A further S-duality turns it into the theory of IIB F1-strings. Note that N measures the lightcone momentum p_+ in the IIA matrix string theory, while it measures the winding number in the IIB matrix string picture.

4.2 Matrix open string coupling to non-Abelian gauge field

Consider a stack of N_p Dp-branes whose worldvolume theory is given by a $\mathrm{U}(N_p)$ gauge theory. We would like to derive the coupling of the worldvolume gauge field A_μ to the matrix open string. Denote the worldvolume coordinates by x^μ, $\mu = 0, 1, \cdots, p$. The gauge field transforms as

$$A_\mu \to U^{-1} A_\mu U + U^{-1} \partial_\mu U. \tag{4.3}$$

We propose a coupling of the form naturally expected for a single string ending on a single Dp-brane

$$\int d\tau \, \mathrm{tr} \left[A_\mu(x^\nu(\tau)) \frac{dX^\mu(\tau)}{d\tau} \right], \tag{4.4}$$

where

$$x^\nu(\tau, \sigma) := \frac{1}{N} \mathrm{tr} X^\nu(\tau, \sigma) \tag{4.5}$$

is the center of mass coordinate of the matrix string coordinates and the sum in (4.4) is over $\mu = 0, 1, \cdots p$.

In the following we make a few comments about this coupling.

1. We note that the above coupling can be made gauge covariant in the following manner. For convenience, we introduce a noncommutativity over the worldvolume

$$[y^\mu, y^\nu] = i\theta^{\mu\nu}, \quad \theta^{\mu\nu} \text{ constant and invertible}, \tag{4.6}$$

and introduce the field

$$C_\mu := -i \left(\theta^{-1} y \right)_\mu + A_\mu(y). \tag{4.7}$$

Unlike A_μ, C_μ transforms covariantly

$$C_\mu \to U^{-1} C_\mu U. \tag{4.8}$$

Note that since $[C_\mu, C_\nu] = i\theta_{\mu\nu}^{-1} + F_{\mu\nu}$, where $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ is the field strength for A_μ,

$$\mathrm{tr} \left[C_\mu, C_\nu \right]^2 = N \theta_{\mu\nu}^{-2} + \mathrm{tr} F_{\mu\nu}^2. \tag{4.9}$$

The first term on the r.h.s. is a constant and can be dropped. Therefore we can use $\mathrm{tr}[C_\mu, C_\nu]^2$ as the Lagrangian and the description in terms of C has a smooth limit as $\theta \to 0$. The coupling (4.4) can be made gauge invariant by replacing A_i with C_i and $dX^i/d\tau$ by $D_\tau X^i$. However, it is the term above which describes the dynamics of the open strings on the D-branes.

2. If we consider precisely the above coupling, there is an obvious problem. In general $N_p \neq N$ so the matrix multiplication and trace don't make sense. However, this is not an issue in the special cases where either $N = 1$ or $N_p = 1$, and it is this latter

case which is of interest to us. However, for completeness we comment on the general case. First, note that we also need appropriate boundary conditions for the (matrix) coordinates $X^{i'}$ orthogonal to the Dp-branes, i.e. strings should actually end on the Dp-branes. This condition should be implemented by requiring the eigenvalues of the $X^{i'}$ at $\sigma = 0$ to be fixed to values corresponding to the position of a Dp-brane (which itself is given by the eigenvalues of scalars in the non-Abelian theory of the N_p Dp-branes.) Intuitively, this is simply the requirement that each string ends on a specific Dp-brane. How many strings end on each Dp-brane is given by a choice of boundary conditions for the system. However, this mapping of N string positions to N_p Dp-brane positions, interpreted as a mapping between the $N \times N$ and $N_p \times N_p$ matrices describing the string and Dp-brane positions, will also provide a way to match the $N \times N$ matrices describing the string positions X^i parallel to the Dp-branes with the $N_p \times N_p$ matrices describing the gauge field on the Dp-branes. However, as stated above we will only explicitly consider the case $N_p = 1$ where the matching is trivial.

3. Finally, we note that the coupling of a single open string to N_p Dp-branes is usually described by inserting a Wilson loop into the standard open string partition function, rather than simply including a boundary coupling in the open string action as we have done here. However, as discussed in [34] these two approaches are related. In fact the description of a single open string ending on a stack on N_p Dp-branes in terms of a Wilson loop is expected to arise in the effective action, after integrating out interactions between the N_p Dp-branes, giving rise to a "fat brane" with an open string ending on it.

4.3 Nahm equation and noncommutative geometry of the D3-brane

Let us now use the IIB matrix string theory as a probe to study the D1-D3 system and to derive the Nahm equation (see figure 2). Since one can identify the endpoint of the open F1-string with the D3-brane worldvolume, so one should be able to understand the Nahm equation as a boundary condition of the open F1-string. In order to identify the worldvolume coordinates of the D3-brane with the F1-string, we need to consider a matrix theory of F1-string based on U(N) matrices.

As we have seen in the last section, a NS B field will modify the (bulk) scalar interaction of the D1-strings matrix theory quite complicatedly. On the other hand, the B-coupling is a much simpler minimal coupling for the F1-string since it is an electric source for the B-field. For matrix strings, the interaction with the B-field is naturally incorporated by the generalization of the usual coupling

$$S_B := \frac{1}{4\pi\alpha'} \int 2\pi\alpha' B_{\mu\nu} \, \mathrm{tr} D_a X^\mu D_b X^\nu \epsilon^{ab}. \tag{4.10}$$

With a constant B-field, the boundary condition of the matrix F1-string is modified to

$$\frac{1}{2\pi\alpha'} \left(\partial_\sigma X^i + 2\pi\alpha' B^i{}_j \partial_\tau X^j \right) = F^{0i} + F_j{}^i \partial_\tau X^j \tag{4.11}$$

at the endpoint $\sigma = \sigma_0$. It is convenient to group the last term with the B-term on the l.h.s. and write

$$\frac{1}{2\pi\alpha'} \left(\partial_\sigma X^i + 2\pi\alpha' \left(F^i{}_j + B^i{}_j\right) \partial_\tau X^j\right) = F^{0i}. \tag{4.12}$$

Note that the l.h.s. is precisely the boundary condition for a F1-string in the presence of a background $F + B$. Since we would like to describe a D3-brane with a constant B-field on it but with no background F_{ij}, we will let $F_{ij} = 0$ and the boundary condition becomes

$$\frac{1}{2\pi\alpha'} \left(\partial_\sigma X^i + 2\pi\alpha' B^i{}_j \partial_\tau X^j\right) = F^{0i}. \tag{4.13}$$

Let us introduce the dual field strength $\tilde{F}$ defined by:

$$F^{\mu\nu} = \frac{\sqrt{-G}}{2}\epsilon^{\mu\nu\alpha\beta}\tilde{F}_{\alpha\beta} = \frac{1}{2\sqrt{-G}}\epsilon_{\mu\nu\alpha\beta}\tilde{F}_{\alpha\beta}. \tag{4.14}$$

Here the convention for the Hodge duality operation is $\epsilon_{0123} = 1 = -G\epsilon^{0123}$. The boundary condition for the F1-strings thus takes the form,

$$\frac{1}{2\pi\alpha'} \left(\partial_\sigma X^i + 2\pi\alpha' B^i{}_j \partial_\tau X^j\right) = \frac{1}{2}\frac{\epsilon_{ijk}}{\sqrt{-G}}\tilde{F}_{jk}, \tag{4.15}$$

For reasons that will become clear later, we have allowed for the possibility that a nontrivial metric $G_{\mu\nu}$ may appear on the D3-brane worldvolume when $B \neq 0$.

To proceed, we need to identify F^{0i} with a configuration of the boundary values of the scalar fields X^i of the F1-strings. We *propose* that

$$\tilde{F}^{ij} = \frac{i}{(2\pi\alpha')^2} \left[X^i, X^j\right], \quad i,j = 1,2,3, \tag{4.16}$$

This is the S-dual of the relation (3.13). This is reasonable as the B-field couples to the F1-strings only through the term (4.10) and there is no modification to the bulk scalar interaction as in (3.16) for the case of D1-strings in a B-field. Since the effect of the B-field has already been taken into account by (4.10), it is reasonable to propose the identification (4.16) from the S-dual statement of (3.13) which holds for D1-branes without B-field.

When there is no B-field, $G_{\mu\nu} = \eta_{\mu\nu}$. On substituting (4.16), the boundary condition (4.15) yields

$$\frac{1}{2\pi\alpha'}\partial_\sigma X^i = \frac{i}{(2\pi\alpha')^2} \left[X^i, X^j\right], \quad i,j = 1,2,3, \tag{4.17}$$

at $\sigma = \sigma_0$. Here X^i are the matrix coordinates of the F1-strings. Since the boundary coordinates can be identified with the D1-branes coordinates, and since the F1-strings can end anywhere on the D1-branes (see figure 2), this equation actually describes the D3-brane and we arrive precisely at the original Nahm equation (2.11).

Next we consider the case with $B \neq 0$. Since turning on B_{12} breaks the Lorentz group $SO(1,3)$ to $SO(1,1) \times SO(2)$, a metric $G_{\mu\nu}$ of the following form is allowed:

$$G_{\mu\nu} = \begin{pmatrix} -g_0 & 0 & 0 & 0 \\ 0 & g_1 & 0 & 0 \\ 0 & 0 & g_2 & 0 \\ 0 & 0 & 0 & g_3 \end{pmatrix}, \tag{4.18}$$

D1 expands as a D3-brane
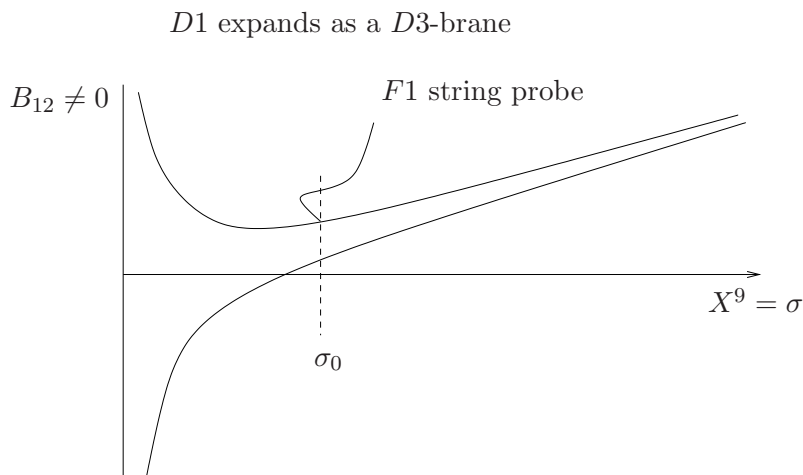
Figure 2. F1-strings probing the D3-brane.

where $g_0 = g_3$ and $g_1 = g_2$. The boundary condition (4.15) reads

$$\partial_\sigma X^i + 2\pi\alpha' B^i_{\ j}\partial_\tau X^j = \frac{i\epsilon_{ijk}}{2\lambda}\frac{g_j g_k}{g_0 g_1}\left[X^j, X^k\right].\tag{4.19}$$

Note that when there is no B-field, the equation (4.19) can be solved by a static configuration. With a B-field turned on, this is no longer the case. A simple ansatz for $X^i(\tau, \sigma)$ is to make an amendment to the static ansatz as follows:

$$X^i(\tau, \sigma) = X_0^i(\tau, \sigma)\mathbf{1} + Y^i(\sigma),\tag{4.20}$$

where X_0^i and Y^i satisfy

$$\partial_\sigma X^i + 2\pi\alpha' B^i_{\ j}\partial_\tau X^j = 0,\tag{4.21}$$

$$\partial_\sigma Y^i = \frac{i\epsilon_{ijk}}{2\lambda}\frac{g_j g_k}{g_0 g_1}\left(\left[X_0^j, X_0^k\right]\mathbf{1} + \left[X_0^j\mathbf{1}, Y^k\right] + \left[Y^j, X_0^k\mathbf{1}\right] + \left[Y^j, Y^k\right]\right)\tag{4.22}$$

at the endpoint of F1-strings.

As was analyzed in [18–20], the mixed boundary condition (4.21) is not compatible with the standard canonical quantization. The proper quantization taking into account the B-field has been carried out in [18–20] with the essential result that the zero modes x_0^i become noncommutative. These results can be applied here immediately as the quantization problem is the same. The result in terms of X_0^i reads

$$\left[X_0^i(\tau, \sigma), X_0^j(\tau, \sigma')\right] = \begin{cases} i\theta^{ij}, & \text{for } \sigma = \sigma' = \sigma_0 \\ 0, & \text{otherwise} \end{cases}\tag{4.23}$$

where $\theta^{ij} = (2\pi\alpha')^2\left(B(1 - (2\pi\alpha')^2 B^2)^{-1}\right)^{ij}$. For the present case $B = B_{12} \neq 0$, it is

$$\theta^{12} = \frac{(2\pi\alpha')^2 B}{(1 + (2\pi\alpha' B)^2)} := \theta, \qquad \text{and} \quad \theta^{ij} = 0, \quad \text{otherwise.}\tag{4.24}$$

– 16 –

The equation (4.23) is precisely saying that X_0^i is becoming noncommutative at the point where the F1-strings touch the D1-branes. Thus X_0^i can be thought of as the coordinates describing the underlying noncommutative D3-brane.

As for equation (4.22), we treat it as a classical equation and so the commutator of Y^i with X_0^j vanishes. We obtain

$$\partial_\sigma Y^i = \frac{i}{2\lambda} \frac{g_j g_k}{g_0 g_1} \epsilon_{ijk} \left(\left[Y^j, Y^k \right] + i\theta \delta_3^i \right). \tag{4.25}$$

With Y^i identified with the D3-brane coordinates, one can think of (4.25) as the equation describing the shape of the D1-branes spike in figure 2. Written explicitly, we have

$$\partial_\sigma Y^i = \frac{i}{\lambda} \epsilon^{ij} \left[Y^j, Y^3 \right], \quad i,j = 1,2,$$
$$\partial_\sigma Y^3 = \frac{i}{\lambda} \frac{g_1}{g_0} \left(\left[Y^1, Y^2 \right] + i\theta \right). \tag{4.26}$$

These are the equations which describe the D3-brane with B-field from the D1-strings point of view.

An interesting result of this analysis is that the constant term θ in (4.26) has a simple interpretation in terms of the noncommutative geometry of the D3-brane: $[X_0^i, X_0^j] = i\theta^{ij}$. With this as an input, we can now fix the metric components g_0 and g_1. Recall that in the presence of a B-field, the worldvolume theory of a D3-brane is given by a noncommutative Yang-Mills theory with noncommutativity parameters $\theta^{\mu\nu}$ and with the open string metric $G_{\mu\nu}$ [17]. In contrast to the closed string metric $g_{\mu\nu}$, $G_{\mu\nu}$ is the effective metric seen by the open strings and is the one relevant for the description of the noncommutative Yang-Mills. In the present case with only $B_{12} \neq 0$ and $g_{\mu\nu} = \eta_{\mu\nu}$, $G_{\mu\nu}$ is given by (4.18) with

$$g_0 = 1, \quad g_1 = 1 + \lambda^2 B^2. \tag{4.27}$$

And our (4.26) reproduces precisely the modified Nahm equation (2.19).

This understanding of the Nahm equation as boundary conditions for the F1-strings has provided us with a more precise understanding of the physical meaning of the B-field modifications in the Nahm equation (2.19) in terms of noncommutative geometry ($\theta^{\mu\nu}$ and $G_{\mu\nu}$) of the open string in a background B-field.

5 C-field modification of the Basu-Harvey equation

Following a similar analysis as for the Nahm equation, we now show that the original Basu-Harvey equation describing an M5-brane can also be understood as a boundary condition of the theory of multiple membranes.

One of the interesting features of the theory of the M5-brane is that the worldvolume field strength H satisfies a nonlinear self-duality condition. In the formulation of [11], the three-form field strength $H_{\mu\nu\lambda}$ on the worldvolume of the M5-brane is expressed in terms of an auxiliary self-dual field strength $h_{abc} = \frac{1}{6}\epsilon_{abcdef}h^{def}$ as

$$H_{\mu\nu\lambda} = E_\mu{}^a E_\nu{}^b E_\lambda{}^c m_b{}^d m_c{}^e h_{ade} = e_\mu{}^a e_\nu{}^b e_\lambda{}^c \left(m^{-1} \right)_c{}^e h_{abe}, \tag{5.1}$$

where

$$m_a{}^b := \delta_a{}^b - 2h_{acd}h^{bcd}, \tag{5.2}$$

The non-linear self-duality condition for H is

$$\frac{\sqrt{-\det g}}{6}\epsilon_{\mu\nu\lambda\gamma\rho\sigma}H^{\gamma\rho\sigma} = \frac{1+K}{2}\left(\mathcal{G}^{-1}\right){}^{\rho}_{\mu}H_{\nu\lambda\rho}, \tag{5.3}$$

where $g_{\mu\nu}$ is the metric on the M5-brane, $K := \sqrt{1 + \frac{1}{24}H_{\mu\nu\lambda}g^{\mu\gamma}g^{\nu\rho}g^{\lambda\sigma}H_{\gamma\rho\sigma}}$. The metric

$$\mathcal{G}_{\mu\nu} = E_\mu{}^a E_\nu{}^b \eta_{ab} \tag{5.4}$$

has its vielbein given by

$$E_\mu{}^a := e_\mu{}^b \left(m^{-1}\right)_b{}^a, \tag{5.5}$$

where $e_\mu{}^a$ is the vielbein of the metric $g_{\mu\nu}$. It turns out this nonlinear self-duality condition will play an important role in our analysis below.

5.1 The Basu-Harvey equation

Let us first review the Basu-Harvey equation which has been proposed to describe the system of multiple M2-branes ending on an M5-brane. Consider the system with N M2-branes lying in the 0,1,9 directions and the M5-brane in the 0,1,2,3,4,5 directions. The Basu-Harvey equation is

$$\partial_2 X^i + i\frac{K}{3!}\epsilon_{ijkl}\left[X^j, X^k, X^l\right] = 0, \quad i = 2, 3, 4, 5. \tag{5.6}$$

Here $(\tau, \sigma_1, \sigma_2)$ are the worldvolume coordinates of the membranes, $X^i(\sigma_2)$ describes the transverse fluctuations to the M2-branes, K is a constant of mass dimension 3 and $\epsilon_{2345} = 1$. In [5], X^i is taken to be valued in an algebra $\mathcal{A}_4$ with generators T^i, $i = 1, \cdots, 4$ and the 3-bracket satisfies the SO(4)-invariant relation

$$\left[T^i, T^j, T^k\right] = i\epsilon_{ijkl}T^l. \tag{5.7}$$

The Basu-Harvey equation is solved by

$$X^i(\sigma_2) = f(\sigma_2)T^i, \tag{5.8}$$

where

$$f(\sigma_2) = \frac{1}{\sqrt{2K\sigma_2}}. \tag{5.9}$$

Note that the operator $C := \sum(T^i)^2$ is central in the algebra (5.7) and so one can consider representations of the algebra (5.7) with constant values of this operator. For each such representation, the solution (5.8) describes a wedge of M2-branes which open up to the M5-brane and whose cross-section is described by a fuzzy three-sphere [39] with radius $r^2(\sigma_2) := \sum(X^i)^2$ given by

$$r^2(\sigma_2) = \frac{C}{2K\sigma_2}. \tag{5.10}$$

By choosing the values of the constants K and C appropriately, one reproduces precisely the radius-to-distance relation obtained from the M2-brane solution [28] solved in the M5-brane field theory

$$r^2(\sigma_2) = \frac{Q}{\sigma_2}. \qquad (5.11)$$

where

$$Q = 3\pi^2 l_P^3 N \qquad (5.12)$$

for the M2-M5 branes system and $l_P = g_s^{1/3} l_s$ is the Planck length. For more general solutions to the Basu-Harvey equation and interpretation in terms of M-branes, see [40].

Just as the Nahm equation can be understood as a BPS equation in the D1-branes Born-Infeld theory, the Basu-Harvey equation can also be understood as a BPS equation in the BL theory for multiple membranes [1–3]. This was originally derived for the BL theory based on the 3-algebra $\mathcal{A}_4$, although the argument applies equally to a BL theory based on a general Lie 3-algebra. However since the inner product for a Lie 3-algebra cannot be positive definite apart from the $\mathcal{A}_4$ case [41], the theory will necessarily contain zero or negative norm states. Therefore it is not clear whether these theories, and the resulting BPS equations, make sense. Attempts have been made to make sense of the BL theory of multiple membranes based on a particular form of Lorentzian 3-algebras [7]. The negative norm state there can be eliminated if one enriches the theory with additional fields such that there is an additional gauge symmetry. The resulting theory however turns out to be equivalent to the D2-brane supersymmetric Yang-Mills theory [8]. Nevertheless we do not rule out the possibility that for a certain specific kind of 3-algebra, the BL theory is physically well defined, perhaps after certain extensions and modifications of the original construction.

Assuming such a theory of (an arbitrary number of) multiple M2-branes exists, the Basu-Harvey equation which describes the M5-brane where M2-branes end is presumably just the BPS condition of this theory. However since we do not have the knowledge of the form of the action as well as the supersymmetry transformations, we will have to take a different route to derive this equation. As we will show next, the Basu-Harvey equation can be understood as the boundary condition of the M2-branes probing the M2-M5 intersecting system. The advantage of this approach is that much less knowledge about the structure of the theory of multiple membranes is needed to derive the boundary condition.

5.2 Basu-Harvey equation as boundary condition of membrane theory

Since eventually we will be interested in adding a 3-form C-field, and since the Born-Infeld type nonlinear generalization of the multiple membrane theory including the effects of the C-field is unknown, let us therefore use use the brane-probe approach to derive the Basu-Harvey equation as the boundary condition of the probing M2-branes.

Let us first consider the case without C-field. To derive the boundary condition of the membrane, we will assume that the action (in the static gauge) contains the kinetic term

$$S = \int d^3\sigma \frac{1}{2}\langle D_a X^I, D_a X^I\rangle + \cdots \qquad (5.13)$$

where $\langle\,,\,\rangle$ denotes an inner product of the 3-algebra and $\cdots$ denotes fermions and gauge field kinetic terms and interaction terms. In the original BL theory, the kinetic term in (5.13) is the only term that contributes to the boundary condition of X^I when the membranes are taken to be open. The $\cdots$ terms do not contribute. We will make the mild assumption that this will continue to be the case in the general theory of membranes.

Now consider a system of multiple membranes extending in the 0,1,9 directions and growing into an M5-brane which extends in the 0,1,2,3,4,5 directions. For a single membrane ending on an M5-brane, the boundary of the membrane appears as a non-gravitational string sourcing the self-dual 2-form $\mathcal{B}$ on the M5-brane

$$\int \mathcal{B}_{\mu\nu}\partial_a X^\mu \partial_b X^\nu \epsilon^{ab}, \qquad \mu,\nu = 0,1,\cdots,5, \quad a,b = \tau,\sigma_1. \tag{5.14}$$

For multiple membranes, the boundary becomes multiple self-dual strings. Most generally, one would like to describe the coupling for multiple self-dual strings to multiple M5-branes, just as the coupling (4.4) for multiple strings with multiple D-branes. However this requires the knowledge of "non-Abelian" tensor multiplets and a representation of the 3-algebra. Without knowing this, we will have to be content with the coupling of multiple self-dual strings to a single M5-brane. This is analogous to the coupling of the center of mass string to a single D-brane $\int A_\mu \dot{x}^\mu$ and can be written down easily

$$\int \mathcal{B}_{\mu\nu}\langle D_a X^\mu, D_b X^\nu\rangle \epsilon^{ab}, \quad a,b = \tau,\sigma_1 \tag{5.15}$$

assuming the membrane is extended in X^1 and has a boundary at $\sigma_2 = 0$. Taking a static gauge $X^0 = \tau, X^1 = \sigma_1$, it is easy to derive the boundary condition for the open membranes. It is

$$\partial_{\sigma_2} X^i + F^i{}_{jk}\partial_\tau X^j \partial_{\sigma_1} X^k = F^{01i} + F^{i0}{}_j \partial_{\sigma_1} X^j + F^{i1}{}_j \partial_\tau X^j, \quad i,j = 2,3,4,5, \tag{5.16}$$

for $\sigma_2 = 0$. Here $F := d\mathcal{B}$ and $H = d\mathcal{B} + C$ is the self-dual 3-form field strength of the M5-brane. Note that the l.h.s. is precisely the combination that will appear in the boundary condition of a membrane in the presence of a background 3-form C-field. In this section we do not consider such a C-field, so let us set the background $F_{ijk} = 0$.

As in the case of F1-strings, it is convenient to introduce the dual field strength $\tilde{F}$ defined by

$$F^{\mu\nu\lambda} := \frac{\sqrt{-G}}{3!}\epsilon^{\mu\nu\lambda\alpha\beta\gamma}\tilde{F}_{\alpha\beta\gamma} = \frac{1}{3!\sqrt{-G}}\epsilon_{\mu\nu\lambda\alpha\beta\gamma}\tilde{F}_{\alpha\beta\gamma}, \tag{5.17}$$

where the Hodge star is taken with the convention $\epsilon_{012345} = 1 = -G\epsilon^{012345}$. We have allowed for the possibility of having a non-trivial metric $G_{\mu\nu}$ on the M5-brane in general. F^{01i} can thus be expressed in terms of the spatial components $\tilde{F}_{jkl}$ of the dual field strength. As for the components $F^{0i}{}_j$ and $F^{1i}{}_j$, we will consider the case where they are equal to zero, which corresponds to a particular configuration of M5-brane. The equation (5.16) now reads

$$\partial_2 X^i = \frac{1}{3!\sqrt{-G}}\epsilon_{ijkl}\tilde{F}_{jkl}, \quad i,j,k,l = 2,3,4,5. \tag{5.18}$$

Repeating the probe argument as for the F1-strings, the M5-brane is defined by the equation (5.18) with an appropriate identification of the field strength $\tilde{F}_{jkl}$ with a certain configuration of the boundary value of X^i. Inserting back properly the dimensional proportional constant K, we propose that

$$\tilde{F}^{ijk} = ifK\left[X^i, X^j, X^k\right], \quad i,j,k = 2,3,4,5, \tag{5.19}$$

where, in general, the scalar f is a function of the background C, and $f = 1$ when $C = 0$. We will make more comment on this relation in the discussion section. For now, let us mention that the relation (5.19) is consistent with the relation (4.16) for the F1-strings.

Back to our analysis. In the case when there is no C-field, the equation (5.18) reads

$$\partial_2 X^i = i\frac{K}{3!}\epsilon_{ijkl}\left[X^j, X^k, X^l\right]. \tag{5.20}$$

when the identification (5.19) is substituted. This is precisely the original Basu-Harvey equation. Here we have derived it as a boundary condition of the open membrane. We note that the identification (5.19) allows us to represent the worldvolume field strength of the M5-brane in terms of a 3-bracket of scalar fields of the M2-brane. It will be interesting to understand better its other implications in the physics of M2-branes and M5-branes. We also remark that when there is no C-field, one can also derive the Basu-Harvey equation as the boundary condition of the original set of M2-branes as in section 3.

We also remark that in general it may be possible to find more general configurations of the M2-branes which also excite the components $F^{0i}{}_j$ and $F^{1i}{}_j$. This will modify the equation (5.18). It would be interesting to understand what would be the identification of these components in terms of X and to understand the resulting equation which generalizes the Basu-Harvey equation; and to study the properties of these more general configurations of M5-branes.

5.3 C-field modification to Basu-Harvey equation and the quantum geometry of the M5-brane

Let us now incorporate a constant C-field on the M5-brane and ask how it modifies the Basu-Harvey equation (5.20).

To start with, we recall that the system of M2-branes ending on M5-branes with a constant C-field ($C_{012}, C_{345} \neq 0$) has been studied from the M5-brane point of view before [29, 30] where the M2-brane was constructed as a soliton of the M5-brane equation of motion. In the papers [29, 30], a constant worldvolume field strength $h_{012} = -h_{345} = h = \frac{1}{2}\tan\frac{\alpha}{2}$ was considered. The components of H and the auxiliary field h are related by $H_{012} = \frac{\sin\alpha}{4}$, $H_{345} = -\frac{\tan\alpha}{4}$. This is equivalent to a background with $F = 0$ and a constant 3-form potential C

$$C_{012} = \frac{\sin\alpha}{4}, \quad C_{345} = -\frac{\tan\alpha}{4} \tag{5.21}$$

due to the tensor-gauge symmetry which keeps $H = C + F$ invariant.

The M2-brane soliton extends in the $X^9 = \sigma_2$ direction and has a cross-section $R \times S_3$ where S_3 is an ellipsoid described by

$$\frac{x_3^2}{r_3(\sigma_2)^2} + \frac{x_4^2}{r_4(\sigma_2)^2} + \frac{x_5^2}{r_5(\sigma_2)^2} + \frac{(x_2 - \sigma_2 \tan \alpha)^2}{r_2(\sigma_2)^2} = 1. \tag{5.22}$$

Here the radii are

$$r_2^2 = \frac{Q}{\sigma_2 \cos \alpha}, \qquad r_3^2 = r_4^2 = r_5^2 = \frac{Q}{\sigma_2} \cos \alpha = r_2^2 \cos^2 \alpha, \tag{5.23}$$

where Q is given by (5.12). This solution can be interpreted as a bundle of N M2-branes coming out of the M5-brane. The σ_2-dependent shift in the x_2 tells us that the M2-branes wedge is tilted away from the normal to the M5-brane with an angle α. Similar to the case of the D1-branes, the amount of tilting is determined by a balance of the pull on the boundary of the M2-branes due to the C-field component C_{012} and the M2-brane tension.

To understand these results in terms of the M2-branes, we propose the following simple modification to the Basu-Harvey equation:

$$\partial_2 \phi^i = \frac{i\beta}{3!} \epsilon_{ijkl} \left[\phi^j, \phi^k, \phi^l \right] - \delta_2^i K^{1/2} \tan \alpha, \tag{5.24}$$

where

$$\beta = \cos \alpha, \tag{5.25}$$

$$\frac{\phi^i}{K^{1/2}} := \begin{cases} (1 + \tan^2 \alpha)^{1/2} X^i, & \text{for } i = 3, 4, 5, \\ X^i, & \text{for } i = 2, \end{cases} \tag{5.26}$$

and $\tan \alpha = 4C$ where $C := -C_{345}$. Here X^i is the variable to be used to match with the M5-brane soliton result reviewed above. The parameter β, the shift in (5.24) and the scaling factors in (5.26) are fixed by matching with the radius r_2, the σ_2-shift in (5.22), and the ratio of radii in (5.23) respectively. To derive this, we use the fact that the equation (5.24) is invariant under SO(4) rotations, implying that a solution which is spherical symmetric when written in ϕ can be constructed. This solution, when rewritten in terms of X, features the ellipsoidal cross sectional geometry and the σ_2-dependent shift as in (5.22).

Next we would like to derive the generalized Basu-Harvey equation (5.24) as a boundary condition of probe open-membranes. We start with a single membrane ending on an M5-brane with a constant worldvolume C-field. The coupling of the single membrane to the C-field is

$$\int d^3\sigma C_{\mu\nu\lambda} \partial_0 X^\mu \partial_1 X^\nu \partial_2 X^\lambda. \tag{5.27}$$

For a membrane with endpoint at $\sigma_2 = \sigma_{20}$, the boundary condition reads

$$\partial_2 X^i + C^i{}_{jk} \partial_0 X^j \partial_1 X^k = \frac{1}{3!\sqrt{-G}} \epsilon_{ijkl} \tilde{H}_{jkl}, \quad i, j, k, l = 2, 3, 4, 5. \tag{5.28}$$

Since our C-field configuration breaks $SO(1,5) \rightarrow SO(1,2) \times SO(3)$, one can expect the metric $G_{\mu\nu}$ to be of the form:

$$G_{\mu\nu} = \begin{pmatrix} -g_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & g_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & g_1 \end{pmatrix} \tag{5.29}$$

For the analysis below, we do not assume any relation between this metric and the metric $\mathcal{G}_{\mu\nu}$ which appears in the self-duality condition of H.

For multiple membranes, the natural generalization to the coupling (5.27) is

$$S_C = \frac{1}{3!} \int d^3\sigma \, C_{\mu\nu\lambda} \text{tr} \left(D_a X^\mu, D_b X^\nu, D_c X^\lambda \right) \epsilon^{abc}, \tag{5.30}$$

where D_a $(a = 0, 1, 2)$ is the covariant derivative on the multiple M2-branes theory. To write down this coupling, we have assumed that the 3-algebra $\mathcal{A}$ is equipped with a map

$$\text{tr} : \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \rightarrow \mathbb{C}, \tag{5.31}$$

which is completely symmetric and is invariant in the following sense:

$$\text{tr}([\alpha, \beta, f], g, h) + \text{tr}([f, [\alpha, \beta, g], h) + \text{tr}(f, g, [\alpha, \beta, h]) = 0. \tag{5.32}$$

In terms of generators, the map tr can be specified by the constants

$$d^{abc} = \text{tr}(T^a, T^b, T^c). \tag{5.33}$$

An explicit construction of the map tr has been given before in [43] for the case that the 3-algebra is given by the Lorentzian 3-algebra [7]. Another example is to take tr equal to the integration $\int d^3 y$ if the (infinite dimensional) 3-algebra is given by the Nambu bracket $[f, g, h] = \epsilon^{ijk} \partial_i f \partial_j g \partial_k h$ over a 3-manifold with local coordinates $y^{1,2,3}$. Such an infinite dimensional 3-algebra has been employed in [44] to construct the M5-brane theory out of the multiple M2-branes theory. In general it is an interesting question to understand which class of 3-algebras admits such a map. It appears to us that this is an essential requirement for a theory of multiple M2-branes since one must be able to incorporate a coupling to the C-field and (5.30) is the most natural candidate of such a coupling. It would be interesting to understand better this mathematical property and how is could be incorporated into the classification of 3-algebras suitable for BL theories [45]. It is possible that one needs a representation of $\mathcal{A}$.

In general, given the map tr, one can construct a linear and symmetrical product $*$ on the 3-algebra

$$* : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \tag{5.34}$$

by

$$f * g = \frac{d^{abc}}{d^{000}} f_a g_b T^c, \quad \text{where } f = f_a T^a, g = g_b T^b, \tag{5.35}$$

where $T^0 = \mathbf{1}$ is the identity operator defined by the property that $[\mathbf{1}, T^b, T^c] = 0$ for all b, c and by the normalization $\langle \mathbf{1}, \mathbf{1} \rangle = 1$. We will assume that $\mathcal{A}$ has such an (unique) identity operator. If we also assume that, just as for Lie algebra, T^0 does not appear on the r.h.s. of a 3-bracket, i.e. $f^{abc}{}_0 = 0$ for all a, b, c, then it is easy to check that the condition (5.32) is solved by $d^{0ab} = d^{000} \delta^a_0 \delta^b_0$. This implies that $\mathbf{1} * \mathbf{1} = \mathbf{1}$. We will consider 3-algebra of this form in the following. With the aid of this product, the boundary condition for the probe M2-branes thus takes the form $(i, j = 2, 3, 4, 5)$,

$$\partial_2 X^i + C^i{}_{jk} \partial_0 X^j * \partial_1 X^k = \frac{1}{3!\sqrt{-G}} \epsilon_{ijkl} G_{jj} G_{kk} G_{ll} \tilde{F}^{jkl} \tag{5.36}$$

for $\sigma_2 = \sigma_{20}$. As before we have set the field strength $F^i{}_{jk}, F^{i0}{}_j, F^{i1}{}_j = 0$ and content ourselves with considering an M5-brane whose shape is described by the components F^{01i} of the field strength.

Substituting the identification (5.19) into (5.36), we arrive at the equation

$$\partial_2 X^i + C_{ijk} \partial_0 X^j * \partial_1 X^k = \frac{ifK}{3!\sqrt{-G}} \epsilon^{ijkl} G_{jj} G_{kk} G_{ll} [X^j, X^k, X^l], \quad i, j = 2, 3, 4, 5. \tag{5.37}$$

This equation is what one would expect to define an M5-brane from the open M2-branes theory. As in the case of the D3-brane, the equation (5.37) reproduces our Basu-Harvey equation (5.24) if we substitute the ansatz

$$X^i(\tau, \sigma_1, \sigma_2) = X^i_0(\tau, \sigma_1, \sigma_2) \mathbf{1} + Y^i(\sigma_2), \tag{5.38}$$

where $X^i_0(\tau, \sigma_1, \sigma_2)$ and $Y^i(\sigma_2)$ satisfy

$$\partial_2 X^i_0 + C_{ijk} \partial_0 X^j_0 \partial_1 X^k_0 = 0, \tag{5.39}$$

$$\partial_2 Y^i = \frac{ifK}{3!\sqrt{-G}} \epsilon^{ijkl} G_{jj} G_{kk} G_{ll} \left(\left[X^j_0, X^k_0, X^l_0 \right] + \left[Y^j, Y^k, Y^l \right] \right) \tag{5.40}$$

at the endpoint of the probe M2-branes. Written more explicitly, this reads

$$\partial_2 Y^2 = ifK \left(\frac{g_1}{g_0} \right)^{3/2} \left(\left[X^3_0, X^4_0, X^5_0 \right] + \left[Y^3, Y^4, Y^5 \right] \right), \tag{5.41}$$

$$\partial_2 Y^i = ifK \left(\frac{g_1}{g_0} \right)^{1/2} \epsilon^{i2jk} \left(\left[X^2_0, X^j_0, X^k_0 \right] + \left[Y^2, Y^j, Y^k \right] \right), \quad i, j, k = 3, 4, 5. \tag{5.42}$$

This agrees exactly with (5.24) if

$$f = \cos \alpha, \tag{5.43}$$

$$\frac{g_1}{g_0} = 1 + \tan^2 \alpha \tag{5.44}$$

and

$$\begin{aligned}
\left[X^2_0, X^j_0, X^k_0 \right] &= 0, \\
\left[X^j_0, X^k_0, X^l_0 \right] &= i\Theta^{jkl}, \quad j, k, l = 3, 4, 5,
\end{aligned} \tag{5.45}$$

where

$$\Theta^{jkl} = \epsilon^{jkl} \frac{1}{K} \frac{C}{(1+C^2)^2}. \tag{5.46}$$

We note that the relation (5.44) is precisely satisfied by the metric (5.4). In fact, for our C-field configuration, we find that the metric $\mathcal{G}_{\mu\nu}$ is of the form of (5.29) with

$$g_0 = \cos^4 \frac{\alpha}{2}, \qquad g_1 = \frac{\cos^4 \frac{\alpha}{2}}{\cos^2 \alpha}. \tag{5.47}$$

It has been postulated in [21] that the metric $\mathcal{G}_{\mu\nu}$ plays the role of the open membrane metric just as the open string metric for D3-brane in background B-field. Our analysis verifies this claim independently.

It is worthwhile to explore more deeply the meaning of the result (5.45). Since the boundary variables X_0^i satisfy precisely the mixed boundary condition of an open membrane ending on an M5-brane with a C-field, one can identify X_0^i as the coordinates of the underlying M5-brane. The variables Y^i should be identified with the M2-brane excitations which describe the protruding M5-brane as an M2-branes wedge. Our result (5.45) implies that the M5-brane worldvolume should satisfy the quantum geometry relations (5.45) for a C-field configuration given by (5.21). The relation (5.45) was obtained with a physical gauge $X_0 = \tau, X^1 = \sigma_1, X^9 = \sigma_2 + \sigma_{20}$. Properly covariantizing the results, we expect the quantum geometry of the M5-brane takes the form

$$\left[X_0^\mu, X_0^\nu, X_0^\lambda \right] = i\Theta^{\mu\nu\lambda}, \tag{5.48}$$

where

$$\Theta^{\mu\nu\lambda} = \begin{cases} \epsilon^{\mu\nu\lambda} \frac{C'}{K(1-C'^2)^2} & \mu, \nu, \lambda = 0, 1, 2, \\ \epsilon^{\mu\nu\lambda} \frac{C}{K(1+C^2)^2} & \mu, \nu, \lambda = 3, 4, 5, \\ 0 & \text{otherwise}, \end{cases} \tag{5.49}$$

and $C := -4C_{345}$, $C' := 4C_{012}$.

The result (5.48) is intriguing. In the literature, there have been attempts [21] to try to deduce the quantum geometry of the M5-brane by following the same logic as in the D-brane case [18] by quantising an open membrane in the presence of a constant C-field. However the analysis is much more complicated due to the nonlinear nature of the membrane action and one can only do an approximate analysis. Since a canonical quantization is carried out, these results were expressed in terms of a non-vanishing commutator of the boundary string coordinate, that is, in terms of a noncommutative geometry. The expression is however quite complicated even with the simplification due to the approximation. Compared to these results, the quantum geometry (5.48) is expressed in terms of the 3-algebra and is much simpler and more elegant. Our result suggests that the correct language to express the quantum geometry of the M5-brane in the presence of a constant C-field is in terms of a 3-bracket, rather than a commutator.

6 Discussions

In this paper, we have shown that the general Nahm equation which describes a D3-brane with worldvolume B-field can be understood as the boundary condition of the matrix F1-string which ends on it. This approach provides a clear physical understanding of the modifications, due to the B-field, of the original Nahm equation: the constant shift in the Nahm equation is due to the noncommutative geometry of the D3-brane, and the scaling of the different components of the Nahm equation is due to the open string metric of the D3-brane. Applying the same idea to the M2-M5 branes intersecting system, we showed that the modified Basu-Harvey equation we proposed can also be understood in terms of the boundary condition of the multiple membranes which end on the M5-brane if the quantum geometry of the M5-brane takes the form (5.48) and if the open-membrane metric on the M5-brane is given by the metric $\mathcal{G}_{\mu\nu}$ which appears in the nonlinear self-duality condition of H. The prediction of the form of the quantum geometry of the M5-brane in the presence of a constant C-field is the main result of this paper.

A crucial step in our proposed identification of the quantum geometry of the M5-brane is the identification (5.19) in the presence of the C-field. This is the analogue of the relation (4.16) for F1-strings probing the D1-D3 system. Now, in (5.19) we had an arbitrary scalar function of the C-field, f, which in the system we considered in section 5.3 was given by $f = \cos\alpha$, so that the relation (5.19) reads

$$\tilde{F}^{ijk} = iK\cos\alpha \left[X^i, X^j, X^k \right]. \tag{6.1}$$

An understanding of this relation can be obtained in the presence of the special configuration (5.21) of the C-field. It is easy to see that in this case the matrices m, and $\mathcal{G}$ which appear in the nonlinear self-duality condition (5.3) of $H = C + F$ are block diagonal of the form (5.29) and hence

$$\tilde{H}_{ijk} = f(h) H_{ijk} \tag{6.2}$$

for some scalar function $f(h)$. If we now substitute $H = C + F$ and expand this relation around the given background C, we obtain

$$\tilde{F}_{ijk} = \cos\alpha \, F_{ijk} + o\left(F^3\right). \tag{6.3}$$

This implies that we want the identification

$$F^{ijk} = iK \left[X^i, X^j, X^k \right] + \cdots, \tag{6.4}$$

where $\cdots$ denotes possible higher order correction terms. We note that the relation (6.4) is indeed what one would expect. This can be seen by considering a dimensional reduction of the M2-M5 system on X^5. In this reduction, it becomes a D2-D4 system with a worldvolume RR 3-form potential $C_{(3)} = C' \, dX^0 dX^1 dX^2$ and a worldvolume NS 2-form potential $B_{(2)} = -C \, dX^3 dX^4$. The D2-D4 system has been studied in [54] and it is found that the D2-brane tilts away from the normal of the D4-brane with an angle α due to the NS B-field. With a further T-duality on X^1, it is easy to check that our Basu-Harvey

equation (5.24) becomes the Nahm equation (2.19) for the resulting D1-D3 system if the 3-bracket is related to the commutator through the relation:

$$\left[X^j, X^k, X^5\right] = \left[X^j, X^k\right] K^{1/2}, \quad j,k = 2, 3, 4. \tag{6.5}$$

Since F^{ij5} is identified with F^{ij} of the D3-brane theory, the relation (6.4) and (6.5) leads to $F^{ij} = \lambda^{-2}[X^i, X^j] + \cdots$. This relation can be mapped to one for the F1-strings using the S-duality map. The S-duality map is nonlinear and in the leading order, it is $S[F] = \tilde{F}$. As a result we obtain precisely (4.16) for F1-strings in the leading order. Hence we expect (6.1) to hold.

Our derivation suggests that both (4.16) and (6.1) could be modified with corrections of higher order in F. This would imply higher order corrections to the Basu-Harvey equation and the Nahm equation, which presumably would describe stringy/M-theory corrections to the BIon description of the D1 and M2 spikes. It would be interesting to work this out in more detail. We also comment that for a more general configuration of C-field, the nonlinear self-duality condition will give rise to a more complicated relation than (6.2). It would be interesting to derive the corresponding identification (6.1). This should give a better understanding of the role of the open membrane metric, as well as a more precise description of both the M5- and M2-branes in C-field backgrounds.

The quantum geometry expressed by (5.48) is intriguing. Before one can explore its physical consequences, it is necessary to understand more precisely the nature of this 3-bracket and to understand how to obtain the result (5.48) from a more fundamental approach. Let us further discuss these issues.

On the first question, we would like to suggest that the 3-bracket is given by a Nambu bracket [46]. Some time ago, Nambu advocated a new form of mechanics based on the Nambu bracket. A natural general formulation of Nambu mechanics was analyzed in [47]. While the usual canonical quantization is suitable for quantising the symplectic structure of Hamiltonian mechanics, the volume preserving feature of the Nambu bracket suggests that it is relevant for the theory of the membrane [48]. We note that (6.5) holds in the classical limit if the 3-bracket is given by a Nambu bracket and if the commutator is given by a Poisson bracket. This strongly suggests that the correct form of 3-algebra to be used in the theory of multiple membranes is given by a quantization of the Nambu bracket [42].

The quantization of the Nambu bracket is however a difficult problem. An interesting proposal using a non-associative algebra was originally considered by Nambu [46]. Deformation quantization was considered in [49], and quantization in terms of cubic matrices was analyzed in [50], see also [51]. Recently a class of Nambu brackets was constructed [42] using a consistent truncation, following the idea of fuzzy sphere construction. An interesting property of the multiple membrane theory based on a quantum Nambu bracket is that the entropy law $N^{3/2}$ for multiple membranes has a natural interpretation [42], further suggesting that the 3-algebra which is relevant for the formulation of the theory of multiple membranes is given by a quantum Nambu bracket.

On the second question, we believe that, just like the case of D-branes, the relation (5.48) can be obtained by quantising the open M2-brane in the presence of the C-field.

However it appears that treating the mixed boundary condition as a constraint and canonically quantising the system may not be the best way to proceed. It may be possible that a different choice of the quantisation variables and a reformulation of the quantization is necessary. The relation of Nambu mechanics to Hamiltonian mechanics has been explored in, for example, [52]. It will be interesting to explore if there is a way to reformulate the M2-brane quantization so that the final results take the compact form (5.48).

Now back to the possible physical consequences of (5.48). In the case of D-branes, given the noncommutative geometry expressed in terms of a non-vanishing commutator, one immediately has a Moyal $*$-product representation of the noncommutative geometry which allows one to construct the noncommutative field theory as a higher derivative non-local deformation of the original theory. This framework has led to much interesting physics, including most notably, the IR/UV mixing effect [53] in noncommutative field theory, which provides a toy model to study nonlocal effects in quantum gravity. For the present case, it will be interesting to understand how the geometry (5.48) can be realized. We think it is rather unlikely that (5.48) can be realized in terms of a deformed $*$-(binary)product. On the other hand, it appears that the content of (5.48) is naturally about a deformation of a ternary operation. It has also been suggested that the M5-brane worldvolume theory should be non-associative [55]. It will be very interesting to construct a physical model which is defined on a quantum space obeying relations of the form (5.48) and study its physical consequences.

It is interesting to study more details of the dimensional reduction of the M2-M5 system to the D2-D4 system. In addition to (6.5), we also get from (5.48)

$$\left[X^0, X^1, X^2\right] = i\frac{C'}{K(1 - C'^2)^2}. \tag{6.6}$$

This relation says that a quantum geometry expressed in terms of a 3-bracket should appear on the D4-brane worldvolume as a result of the presence of the RR 3-form $C_{(3)}$. We can also dimensionally reduce the M2-M5 system on the X^1 direction. In this case, the system becomes a tilted F1-string ending on a D4-brane with worldvolume RR 3-form potential $C_{(3)} = -C\,dX^3dX^4dX^5$ and a worldvolume NS 2-form potential $B_{(2)} = C'\,dX^0dX^2$. Among other things, we obtain from (5.48) this time the relation

$$\left[X^3, X^4, X^5\right] = i\frac{C}{K(1 + C^2)^2}. \tag{6.7}$$

Again a quantum geometry of the same form as (6.6) appears due to a RR 3-form potential. It is known that RR-backgrounds can give rise to nontrivial quantum geometry in the form of nonanticommutativity [56]. More recently nonanticommutative geometry due to a RR 4-form potential [57] and its AdS/CFT dual has been studied [58]. We emphasis our relations in terms of a 3-bracket are different and provide a new kind of quantum geometry due to RR-potentials. These results are very intriguing. Since D-branes are much more under control than M-branes, by studying the quantization of the D4 system in the presence of a RR 3-form, it may be possible to derive the relations (6.6) and (6.7) rigorously, and in turn provide us with an understanding of the nature of the 3-bracket.

The proper understanding of the 3-algebraic structure of the quantum geometry (5.48) should help us to understand and construct the theory of the "non-Abelian" tensor multiplet on multiple M5-branes. In the case of D-branes, the essential algebraic structure, Lie-algebra, for the construction of non-Abelian gauge theory for multiple D-branes is the same as in the theory of a single D-brane, though in the presence of a B-field. We are optimistic that something similar is true for the M5-brane(s).

Finally we remark that a piece of nontrivial information about the 3-bracket can be obtained by combining our result (5.48) with an uncertainty relation proposed in [55] for the M5-brane worldvolume in the limit of large C-field. The proposed relation takes the form (in our convention)

$$\delta X^3 \delta X^4 \delta X^5 \sim \frac{1}{KC}. \tag{6.8}$$

It is natural to expect that the uncertainty relation (6.8) follows from the quantum geometry described by (5.48). If this is the case, then (6.8) and (5.48) are in agreement if the 3-bracket obeys the scaling

$$[\cdot, \cdot, \cdot] \sim o\left(C^2\right) \tag{6.9}$$

in the large C-field limit. This can be seen by introducing the rescaled variables $Z^i = X^i(KC)^{1/3}$ and the rescaled 3-bracket $[\cdot, \cdot, \cdot]' = \frac{1}{C^2}[\cdot, \cdot, \cdot]$, then our relation (5.48) can be written as $[Z^3, Z^4, Z^5]' = i$. Now if $[\cdot, \cdot, \cdot]'$ is independent of C, and if (6.8) does follow from (5.48), then the C-dependence in (6.8) is reproduced. We note that this kind of nontrivial dependence on the B-field does not occur for the $*$-commutator in the case of noncommutative geometry of D-branes. The scaling limit (6.9) provides a constraint on the C-field deformation of the 3-bracket, or, on the 3-bracket itself if the remark in the previous paragraph is true.

Acknowledgments

References

[1] J. Bagger and N. Lambert, *Modeling multiple M2's*, *Phys. Rev.* **D 75** (2007) 045020 [hep-th/0611108] [SPIRES].

[2] J. Bagger and N. Lambert, *Gauge symmetry and supersymmetry of multiple M2-branes*, *Phys. Rev.* **D 77** (2008) 065008 [arXiv:0711.0955] [SPIRES].

[3] J. Bagger and N. Lambert, *Comments on multiple M2-branes*, *JHEP* **02** (2008) 105 [arXiv:0712.3738] [SPIRES].

[4] A. Gustavsson, *Algebraic structures on parallel M2-branes*, *Nucl. Phys.* **B 811** (2009) 66 [arXiv:0709.1260] [SPIRES].

[5] A. Basu and J.A. Harvey, *The M2-M5 brane system and a generalized Nahm's equation*, *Nucl. Phys.* **B 713** (2005) 136 [hep-th/0412310] [SPIRES].

[6] N. Lambert and D. Tong, *Membranes on an orbifold*, *Phys. Rev. Lett.* **101** (2008) 041602 [arXiv:0804.1114] [SPIRES];
J. Distler, S. Mukhi, C. Papageorgakis and M. Van Raamsdonk, *M2-branes on M-folds*, *JHEP* **05** (2008) 038 [arXiv:0804.1256] [SPIRES].

[7] J. Gomis, G. Milanesi and J.G. Russo, *Bagger-Lambert theory for general lie algebras*, *JHEP* **06** (2008) 075 [arXiv:0805.1012] [SPIRES];
S. Benvenuti, D. Rodriguez-Gomez, E. Tonni and H. Verlinde, $\mathcal{N} = 8$ *superconformal gauge theories and M2 branes*, *JHEP* **01** (2009) 078 [arXiv:0805.1087] [SPIRES];
P.-M. Ho, Y. Imamura and Y. Matsuo, *M2 to D2 revisited*, *JHEP* **07** (2008) 003 [arXiv:0805.1202] [SPIRES].

[8] M.A. Bandres, A.E. Lipstein and J.H. Schwarz, *Ghost-free superconformal action for multiple M2-branes*, *JHEP* **07** (2008) 117 [arXiv:0806.0054] [SPIRES];
J. Gomis, D. Rodriguez-Gomez, M. Van Raamsdonk and H. Verlinde, *Supersymmetric Yang-Mills Theory From Lorentzian Three- Algebras*, *JHEP* **08** (2008) 094 [arXiv:0806.0738] [SPIRES];
B. Ezhuthachan, S. Mukhi and C. Papageorgakis, *D2 to D2*, *JHEP* **07** (2008) 041 [arXiv:0806.1639] [SPIRES].

[9] O. Aharony, O. Bergman, D.L. Jafferis and J. Maldacena, $\mathcal{N} = 6$ *superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, *JHEP* **10** (2008) 091 [arXiv:0806.1218] [SPIRES].

[10] J. Bagger and N. Lambert, *Three-algebras and $\mathcal{N} = 6$ Chern-Simons gauge theories*, *Phys. Rev.* **D 79** (2009) 025002 [arXiv:0807.0163] [SPIRES].

[11] P.S. Howe and E. Sezgin, $D = 11$, $p = 5$, *Phys. Lett.* **B 394** (1997) 62 [hep-th/9611008] [SPIRES];
P.S. Howe, E. Sezgin and P.C. West, *Covariant field equations of the M-theory five-brane*, *Phys. Lett.* **B 399** (1997) 49 [hep-th/9702008] [SPIRES].

[12] M. Aganagic, J. Park, C. Popescu and J.H. Schwarz, *World-volume action of the M-theory five-brane*, *Nucl. Phys.* **B 496** (1997) 191 [hep-th/9701166] [SPIRES].

[13] P. Pasti, D.P. Sorokin and M. Tonin, *Covariant action for a $D = 11$ five-brane with the chiral field*, *Phys. Lett.* **B 398** (1997) 41 [hep-th/9701037] [SPIRES];
I.A. Bandos et al., *Covariant action for the super-five-brane of M-theory*, *Phys. Rev. Lett.* **78** (1997) 4332 [hep-th/9701149] [SPIRES].

[14] C.S. Chu and E. Sezgin, *M-fivebrane from the open supermembrane*, *JHEP* **12** (1997) 001 [hep-th/9710223] [SPIRES].

[15] A. Connes, M.R. Douglas and A.S. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *JHEP* **02** (1998) 003 [hep-th/9711162] [SPIRES].

[16] M.R. Douglas and C.M. Hull, *D-branes and the noncommutative torus*, *JHEP* **02** (1998) 008 [hep-th/9711165] [SPIRES].

[17] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *JHEP* **09** (1999) 032 [hep-th/9908142] [SPIRES].

[18] C.-S. Chu and P.-M. Ho, *Noncommutative open string and D-brane*, *Nucl. Phys.* **B 550** (1999) 151 [hep-th/9812219] [SPIRES].

[19] C.-S. Chu and P.-M. Ho, *Constrained quantization of open string in background B field and noncommutative D-brane*, *Nucl. Phys.* **B 568** (2000) 447 [hep-th/9906192] [SPIRES].

[20] C.-S. Chu, *Noncommutative open string: neutral and charged*, hep-th/0001144 [SPIRES].

[21] E. Bergshoeff, D.S. Berman, J.P. van der Schaar and P. Sundell, *A noncommutative M-theory five-brane*, *Nucl. Phys.* **B 590** (2000) 173 [hep-th/0005026] [SPIRES];
S. Kawamoto and N. Sasakura, *Open membranes in a constant C-field background and noncommutative boundary strings*, *JHEP* **07** (2000) 014 [hep-th/0005123] [SPIRES].

[22] C.G. Callan and J.M. Maldacena, *Brane dynamics from the Born-Infeld action*, *Nucl. Phys.* **B 513** (1998) 198 [hep-th/9708147] [SPIRES].

[23] N.R. Constable, R.C. Myers and O. Tafjord, *The noncommutative bion core*, *Phys. Rev.* **D 61** (2000) 106009 [hep-th/9911136] [SPIRES].

[24] N.R. Constable, R.C. Myers and O. Tafjord, *Fuzzy funnels: non-abelian brane intersections*, hep-th/0105035 [SPIRES].

[25] S. Moriyama, *Noncommutative monopole from nonlinear monopole*, *Phys. Lett.* **B 485** (2000) 278 [hep-th/0003231] [SPIRES].

[26] R.C. Myers, *Dielectric-branes*, *JHEP* **12** (1999) 022 [hep-th/9910053] [SPIRES].

[27] J.L. Karczmarek and C.G. Callan Jr., *Tilting the noncommutative bion*, *JHEP* **05** (2002) 038 [hep-th/0111133] [SPIRES].

[28] P.S. Howe, N.D. Lambert and P.C. West, *The self-dual string soliton*, *Nucl. Phys.* **B 515** (1998) 203 [hep-th/9709014] [SPIRES].

[29] Y. Michishita, *The M2-brane soliton on the M5-brane with constant 3-form*, *JHEP* **09** (2000) 036 [hep-th/0008247] [SPIRES].

[30] D. Youm, *BPS solitons in M5-brane worldvolume theory with constant three-form field*, *Phys. Rev.* **D 63** (2001) 045004 [hep-th/0009082] [SPIRES].

[31] D.-E. Diaconescu, *D-branes, monopoles and Nahm equations*, *Nucl. Phys.* **B 503** (1997) 220 [hep-th/9608163] [SPIRES].

[32] A. Kapustin and S. Sethi, *The Higgs branch of impurity theories*, *Adv. Theor. Math. Phys.* **2** (1998) 571 [hep-th/9804027] [SPIRES].
D. Tsimpis, *Nahm equations and boundary conditions*, *Phys. Lett.* **B 433** (1998) 287 [hep-th/9804081] [SPIRES].

[33] M.R. Garousi, *On non-linear action of multiple M2-branes*, *Nucl. Phys.* **B 809** (2009) 519 [arXiv:0809.0985] [SPIRES];
R. Iengo and J.G. Russo, *Non-linear theory for multiple M2 branes*, *JHEP* **10** (2008) 030 [arXiv:0808.2473] [SPIRES].

[34] F. Lizzi, N.E. Mavromatos and R.J. Szabo, *Matrix σ-models for multi D-brane dynamics*, *Mod. Phys. Lett.* **A 13** (1998) 829 [hep-th/9711012] [SPIRES].

[35] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, *M theory as a matrix model: a conjecture*, *Phys. Rev.* **D 55** (1997) 5112 [hep-th/9610043] [SPIRES].

[36] L. Motl, *Proposals on nonperturbative superstring interactions*, hep-th/9701025 [SPIRES].

[37] T. Banks, N. Seiberg and S.H. Shenker, *Branes from matrices*, *Nucl. Phys.* **B 490** (1997) 91 [hep-th/9612157] [SPIRES].

[38] R. Dijkgraaf, E.P. Verlinde and H.L. Verlinde, *Matrix string theory*, *Nucl. Phys.* **B 500** (1997) 43 [`hep-th/9703030`] [SPIRES].

[39] Z. Guralnik and S. Ramgoolam, *On the polarization of unstable D0-branes into non-commutative odd spheres*, *JHEP* **02** (2001) 032 [`hep-th/0101001`] [SPIRES].

[40] D.S. Berman and N.B. Copland, *A note on the M2-M5 brane system and fuzzy spheres*, *Phys. Lett.* **B 639** (2006) 553 [`hep-th/0605086`] [SPIRES];
C. Krishnan and C. Maccaferri, *Membranes on Calibrations*, *JHEP* **07** (2008) 005 [`arXiv:0805.3125`] [SPIRES].

[41] J.P. Gauntlett and J.B. Gutowski, *Constraining maximally supersymmetric membrane actions*, `arXiv:0804.3078` [SPIRES];
G. Papadopoulos, *M2-branes, 3-lie algebras and plucker relations*, *JHEP* **05** (2008) 054 [`arXiv:0804.2662`] [SPIRES].

[42] C.-S. Chu, P.-M. Ho, Y. Matsuo and S. Shiba, *Truncated Nambu-Poisson bracket and entropy formula for multiple membranes*, *JHEP* **08** (2008) 076 [`arXiv:0807.0812`] [SPIRES].

[43] M. Li and T. Wang, *M2-branes coupled to antisymmetric fluxes*, *JHEP* **07** (2008) 093 [`arXiv:0805.3427`] [SPIRES].

[44] P.-M. Ho and Y. Matsuo, *M5 from M2*, *JHEP* **06** (2008) 105 [`arXiv:0804.3629`] [SPIRES];
P.-M. Ho, Y. Imamura, Y. Matsuo and S. Shiba, *M5-brane in three-form flux and multiple M2-branes*, *JHEP* **08** (2008) 014 [`arXiv:0805.2898`] [SPIRES].

[45] J.M. Figueroa-O'Farrill, *Three lectures on 3-algebras*, `arXiv:0812.2865` [SPIRES];
P. de Medeiros, J. Figueroa-O'Farrill, E. Mendez-Escobar and P. Ritter, *Metric 3-Lie algebras for unitary Bagger-Lambert theories*, `arXiv:0902.4674` [SPIRES]; *On the Lie-algebraic origin of metric 3-algebras*, `arXiv:0809.1086` [SPIRES];
P. de Medeiros, J.M. Figueroa-O'Farrill and E. Mendez-Escobar, *Metric lie 3-algebras in Bagger-Lambert theory*, *JHEP* **08** (2008) 045 [`arXiv:0806.3242`] [SPIRES]; *Lorentzian Lie 3-algebras and their Bagger-Lambert moduli space*, *JHEP* **07** (2008) 111 [`arXiv:0805.4363`] [SPIRES].

[46] Y. Nambu, *Generalized Hamiltonian dynamics*, *Phys. Rev.* **D 7** (1973) 2405 [SPIRES].

[47] L. Takhtajan, *On foundation of the generalized Nambu mechanics (second version)*, *Commun. Math. Phys.* **160** (1994) 295 [`hep-th/9301111`] [SPIRES].

[48] J. Hoppe, *On M-algebras, the quantisation of Nambu-Mechanics and volume preserving diffeomorphisms*, *Helv. Phys. Acta* **70** (1997) 302 [`hep-th/9602020`] [SPIRES];
D. Minic, *M-theory and deformation quantization*, `hep-th/9909022` [SPIRES];
L.M. Baker and D.B. Fairlie, *Hamilton-Jacobi equations and brane associated Lagrangians*, *Nucl. Phys.* **B 596** (2001) 348 [`hep-th/0003048`] [SPIRES];
Y. Matsuo and Y. Shibusa, *Volume preserving diffeomorphism and noncommutative branes*, *JHEP* **02** (2001) 006 [`hep-th/0010040`] [SPIRES];
B. Pioline, *Comments on the topological open membrane*, *Phys. Rev.* **D 66** (2002) 025010 [`hep-th/0201257`] [SPIRES];
T.L. Curtright and C.K. Zachos, *Deformation quantization of superintegrable systems and Nambu mechanics*, *New J. Phys.* **4** (2002) 83 [`hep-th/0205063`] [SPIRES];
T. Curtright and C.K. Zachos, *Classical and quantum Nambu mechanics*, *Phys. Rev.* **D 68** (2003) 085001 [`hep-th/0212267`] [SPIRES].

[49] G. Dito, M. Flato, D. Sternheimer and L. Takhtajan, *Deformation quantization and Nambu mechanics*, *Commun. Math. Phys.* **183** (1997) 1 [`hep-th/9602016`] [SPIRES].

[50] H. Awata, M. Li, D. Minic and T. Yoneya, *On the quantization of Nambu brackets*, *JHEP* **02** (2001) 013 [`hep-th/9906248`] [SPIRES].

[51] Y. Kawamura, *Structure of generalized matrix mechanics*, *Prog. Theor. Phys.* **109** (2003) 1 [`hep-th/0206184`] [SPIRES]; *Cubic matrix, Nambu mechanics and beyond*, *Prog. Theor. Phys.* **109** (2003) 153 [`hep-th/0207054`] [SPIRES].

[52] F. Bayen and M. Flato, *Remarks concerning Nambu's generalized mechanics*, *Phys. Rev.* **D 11** (1975) 3049 [SPIRES]; N. Mukunda and G. Sudarshan, *Relation between Nambu and Hamiltonian mechanics*, *Phys. Rev.* **D 13** (1976) 2846 [SPIRES].

[53] S. Minwalla, M. Van Raamsdonk and N. Seiberg, *Noncommutative perturbative dynamics*, *JHEP* **02** (2000) 020 [`hep-th/9912072`] [SPIRES].

[54] K. Hashimoto and T. Hirayama, *Branes and BPS configurations of noncommutative/commutative gauge theories*, *Nucl. Phys.* **B 587** (2000) 207 [`hep-th/0002090`] [SPIRES]; S. Moriyama, *Noncommutative monopole from nonlinear monopole*, *Phys. Lett.* **B 485** (2000) 278 [`hep-th/0003231`] [SPIRES].

[55] P.-M. Ho and Y. Matsuo, *A toy model of open membrane field theory in constant 3- form flux*, *Gen. Rel. Grav.* **39** (2007) 913 [`hep-th/0701130`] [SPIRES].

[56] H. Ooguri and C. Vafa, *The C-deformation of gluino and non-planar diagrams*, Adv. Theor. Math. Phys. **7** (2003) 53 [`hep-th/0302109`] [SPIRES]; *Gravity induced C-deformation*, Adv. Theor. Math. Phys. **7** (2004) 405 [`hep-th/0303063`] [SPIRES]; J. de Boer, P.A. Grassi and P. van Nieuwenhuizen, *Non-commutative superspace from string theory*, *Phys. Lett.* **B 574** (2003) 98 [`hep-th/0302078`] [SPIRES]; N. Seiberg, *Noncommutative superspace, $N = 1/2$ supersymmetry, field theory and string theory*, *JHEP* **06** (2003) 010 [`hep-th/0305248`] [SPIRES]; N. Berkovits and N. Seiberg, *Superstrings in graviphoton background and $N = 1/2 + 3/2$ supersymmetry*, *JHEP* **07** (2003) 010 [`hep-th/0306226`] [SPIRES].

[57] S. Ferrara, E. Ivanov, O. Lechtenfeld, E. Sokatchev and B. Zupnik, *Non-anticommutative chiral singlet deformation of $N = (1, 1)$ gauge theory*, *Nucl. Phys.* **B 704** (2005) 154 [`hep-th/0405049`] [SPIRES]; K. Ito and S. Sasaki, *Non(anti)commutative $N = 2$ supersymmetric gauge theory from superstrings in the graviphoton background*, *JHEP* **11** (2006) 004 [`hep-th/0608143`] [SPIRES]; K. Ito, Y. Kobayashi and S. Sasaki, *Deformation of $N = 4$ super Yang-Mills theory in graviphoton background*, *JHEP* **04** (2007) 011 [`hep-th/0612267`] [SPIRES].

[58] C.-S. Chu, S.-H. Dai and D.J. Smith, *AdS/CFT Duality for non-anticommutative supersymmetric gauge theory*, *JHEP* **05** (2008) 029 [`arXiv:0803.0895`] [SPIRES].